A simple representation of max-stable processes

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EVA, August 15-19 2005, Gothenburg, Sweden

MOTIVATING EXAMPLE (Brown and Resnick, 1977).

Consider:

- the sequence of processes

$$\{M_n(s)\}_{s\geq 0} := \left\{ \bigvee_{i=1}^{n} b_n \left(O_i \left(s/b_n^2 \right) - b_n \right) \right\}_{s\geq 0}$$

  where $O_1, O_2, \cdots$ are i.i.d. copies of the Ornstein-Uhlenbeck process; $b_n = (2\log n - \log\log n - \log 4\pi)^{1/2}$

- $\{X_i\}_{i=1}^{\infty}$ an enumeration of the points of a realization of the poisson point process on $\mathbb{R}_+$ with mean measure $x^{-2}dx$

- an i.i.d. sequence of Brownian motions W_i independent of the point process

- $$\{\eta(s)\}_{s\geq 0} \stackrel{d}{=} \left\{ \bigvee_{i=1}^{\infty} X_i \, e^{W_i(s)-s/2} \right\}_{s\geq 0} .$$

Then we have

$$\{M_n(s)\}_{s\geq 0} \stackrel{d}{\to} \{\eta(s)\}_{s\geq 0}$$

in $C(\mathbb{R}_+)$.

POISSON POINT PROCESS

A poisson point process in a state space E, with mean measure μ ($\mu(A) < \infty$ for compact sets A), i.e. PPP(E,μ), is the random counting measure N satisfying

1. $$P(N(A) = k) = \begin{cases} e^{-\mu(A)} \frac{(\mu(A))^k}{k!} & , \mu(A) < \infty \\ 0 & , \mu(A) = \infty \end{cases}$$

   ($k \geq 0$, A Borel set of E)

2. if $A_1, \ldots, A_m$ are mutually disjoint Borel sets of E then $N(A_1), \ldots, N(A_m)$ are independent r.v.'s.

$\{X_i\}_{i=1}^{\infty}$ an enumeration of the points of a realization of the PPP(E,μ), or PPP(E,$f(x)\,dx$) if $\mu(A) = \int_A f(x)\,dx$.

- $C(S)$ - the space of continuous functions on a compact metric space S (e.g. $C([0,1])$)

- sup-norm - $|f|_\infty := \sup_{s\in S} |f(s)|$

- $C^+(S) := \{f \in C(S) : f > 0\}$

SIMPLE MAX-STABLE PROCESS

A stoch. proc. $\{\eta(s)\}_{s\in S}$ in $C^+(S)$ with non-degenerate marginals is simple max-stable if

1. $$\left\{\frac{1}{n}\bigvee_{i=1}^{n} \eta_i(s)\right\}_{s\in S} \stackrel{d}{=} \{\eta(s)\}_{s\in S}$$

   for all positive integers n and η_i i.i.d. copies of η

2. $$P\left(\eta(s) \le x\right) = e^{-1/x}, \quad x > 0,\ s \in S.$$

Technical remark: $C^+(S)$ is not CSMS. Let

$$C_1^+(S) := \{f \in C(S) : f \ge 0,\ |f|_\infty = 1\}$$

and enlarge the space $C^+(S)$ to

$$(0,\infty] \times C_1^+(S)$$

which, with an appropriate metric, is CSMS (de Haan and Lin (2001)).

Proposition 1 (Giné, Hahn and Vatan (1990)). *Let η be a simple max-stable process in $C^+(S)$. There exists a finite Borel measure ρ on*

$$C_1^+(S) := \{f \in C(S) : f \geq 0, |f|_\infty = 1\}$$

with the property that

$$\int_{C_1^+} f(s)\, d\rho(f) = 1 \text{ for all } s \in S\ ,$$

such that

$$\{\eta(s)\}_{s\in S} \stackrel{d}{=} \bigvee_{i=1}^{\infty} \{R_i\ \pi_i(s)\}_{s\in S}$$

where (R_i, π_i) are the points of a PPP($(0,\infty] \times C_1^+(S)$, $r^{-2}\, dr \times d\rho$).

Conversely each process with this representation is simple max-stable.

Theorem 1 (Main result). *Let the process η be simple max-stable in $C^+(S)$. Let $\{R_i\}_{i=1}^{\infty}$ be the points of the PPP($(0,\infty]$, $r^{-2}\,dr$). We can find i.i.d. stoch. proc. $V, V_1, V_2, \cdots$ in $C^+(S)$ with*

$$\begin{aligned} EV(s) &= 1, \quad \textit{for } s \in S, \\ E \sup_{s\in S} V(s) &< \infty \end{aligned} \tag{1}$$

such that

$$\eta \stackrel{d}{=} \bigvee_{i=1}^{\infty} R_i V_i \ . \tag{2}$$

Conversely each process with this representation is simple max-stable.

The process V can be chosen is such a way that

$$\sup_{s\in S} V(s) = c > 0 \quad a.s. \tag{3}$$

Example. Consider $\{(X_i, Y_i)\}$, points of the PPP$(\mathbb{R}^2 \setminus \{(0,0)\},\ (x^2+y^2)^{-3/2}\, dx\, dy)$.

The simple max-stable process

$$\left\{ \frac{1}{2} \bigvee_{i=1}^{\infty} X_i \cos\theta + Y_i \sin\theta \right\}_{0\le\theta\le 2\pi}$$

has the above representation. With

$$x = r\cos\phi$$
$$y = r\sin\phi$$

we have, for $r > 0$, $\phi \in [0, 2\pi]$,

$$(x^2+y^2)^{-3/2}\, dx\, dy = r^{-2}\, dr\, d\phi.$$

Write

$$X_i = R_i \cos\Phi_i$$
$$Y_i = R_i \sin\Phi_i.$$

Since, for each θ, the half plane $\{(x,y) : x\cos\theta + y\sin\theta > 0\}$ contains infinitely many points of the PPP,

$$\begin{aligned} &\frac{1}{2} \bigvee_{i=1}^{\infty} X_i \cos\theta + Y_i \sin\theta \\ &= \frac{1}{2} \bigvee_{i=1}^{\infty} R_i \cos\Phi_i \cos\theta + R_i \sin\Phi_i \sin\theta \\ &= \frac{1}{2} \bigvee_{i=1}^{\infty} R_i \left((\cos\Phi_i \cos\theta + \sin\Phi_i \sin\theta) \vee 0 \right). \end{aligned}$$

Hence the product of points of a PPP$((0,\infty] \times C_1^+([0,2\pi]),\ r^{-2}\, dr \times d\phi)$.

Example (cont.) We had

$$\left\{ \frac{1}{2} \bigvee_{i=1}^{\infty} X_i \cos\theta + Y_i \sin\theta \right\}_{0 \le \theta \le 2\pi}$$

$$= \left\{ \frac{1}{2} \bigvee_{i=1}^{\infty} R_i((\cos\Phi_i \cos\theta + \sin\Phi_i \sin\theta) \vee 0) \right\}_{0 \le \theta \le 2\pi}$$

and in fact

$$\frac{1}{2}((\cos\Phi_i \cos\theta + \sin\Phi_i \sin\theta) \vee 0)$$

are i.i.d. stoch. proc. in $C^+([0, 2\pi])$, for each θ

$$E\left(\frac{1}{2}\cos(\Phi - \theta) \vee 0\right) = \int_{\theta-\pi/2}^{\theta+\pi/2} \frac{1}{2}\cos(\phi - \theta)\, d\phi = 1$$

and

$$E \sup_{0 \le \theta \le 2\pi} (\cos(\Phi - \theta) \vee 0)$$

$$= \int_{\theta-\pi/2}^{\theta+\pi/2} \sup_{0 \le \theta \le 2\pi} \cos(\phi - \theta)\, d\phi = \pi < \infty.$$

PROOF OF THE THEOREM

- η simple max-stable (Giné, Hahn and Vatan):

$$\{\eta(s)\}_{s\in S} \stackrel{d}{=} \bigvee_{i=1}^{\infty} \{R_i \; \pi_i(s)\}_{s\in S}$$

PPP($(0,\infty] \times C_1^+(S)$, $r^{-2}\, dr \times d\rho$)

- $\{\tilde{R}_i, \tilde{\pi}_i(s)\}_{s\in S}$

  PPP($(0,\infty]\times C_1^+(S)$, $\rho(C_1^+)\, r^{-2}\, dr \times d\rho/\rho(C_1^+)$)

$$\{\eta(s)\}_{s\in S} \stackrel{d}{=} \bigvee_{i=1}^{\infty} \{R_i \; \pi_i(s)\}_{s\in S} \stackrel{d}{=} \bigvee_{i=1}^{\infty} \{\tilde{R}_i \; \tilde{\pi}_i(s)\}_{s\in S}$$

- $$\begin{cases} \tilde{\tilde{R}}_i := \tilde{R}_i/\rho(C_1^+) \\ \tilde{\tilde{\pi}}_i := \tilde{\pi}_i \; \rho(C_1^+) \end{cases}$$

  PPP($(0,\infty] \times C_\rho^+$, $r^{-2}\, dr \times dQ$)

  $C_\rho^+ := \{f \in C(S) : f \geq 0, |f|_\infty = \rho(C_1^+)\}$

  dQ is a probability measure

$$\{\eta(s)\}_{s\in S} \stackrel{d}{=} \bigvee_{i=1}^{\infty} \{\tilde{R}_i \; \tilde{\pi}_i(s)\}_{s\in S} \stackrel{d}{=} \bigvee_{i=1}^{\infty} \{\tilde{\tilde{R}}_i \; \tilde{\tilde{\pi}}_i(s)\}_{s\in S}$$

Theorem 2. *Let $X, X_1, X_2, \ldots$ be i.i.d stoch. proc. in $C(S)$. Let $a_s(n)$ positive and $b_s(n)$ real, be continuous functions, $\{Y(s)\}_{s\in S}$ be a stochastic process in $C(S)$,*

$$F_s(x) := P(X(s) \le x)$$

continuous in x and U_s be the left-continuous inverse of $1/(1-F_s)$. The following statements are equivalent:

1.
$$\left\{\max_{i\le n} \frac{X_i(s) - b_s(n)}{a_s(n)}\right\}_{s\in S} \xrightarrow{d} \{Y(s)\}_{s\in S}$$
in $C(S)$ where $a_s(n)$ and $b_s(n)$ are chosen such that $-\log P(Y(s) \le x) = (1 + \gamma(s)x)^{-1/\gamma(s)}$ for all x with $1 + \gamma(s)x > 0$.

2.
$$\left\{\max_{i\le n} \frac{1}{n(1 - F_s(X_i(s)))}\right\}_{s\in S} \xrightarrow{d} \{Z(s)\}_{s\in S} \qquad (4)$$
in $C(S)$ for all $s \in S$, and
$$\lim_{n\to\infty} \frac{U_s(nu) - b_s(n)}{a_s(n)} = \frac{u^{\gamma(s)} - 1}{\gamma(s)} \qquad (5)$$
uniformly for $s \in S$.

The relation between Y and Z is: $\{Z(s)\}_{s\in S} =^d \{(1 + \gamma(s)Y(s))^{1/\gamma(s)}\}_{s\in S}$.

References

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