Extreme Value Theory: An Introduction
by Laurens de Haan and Ana Ferreira

This document contains corrections of errors or mistakes found in the book after publication. We acknowledge the contribution of the readers.
We also give extensions of some of the material given in the book.
This file was last updated on April 11, 2007.

Chapter 1: Limit Distributions and Domains of Attraction

Page 12, line 14, correction: \( b'_n = (2\log n)^{1/2} - \frac{\log \log n + \log(4\pi)}{2(2\log n)^{1/2}} \).

Chapter 2: Extreme and Intermediate Order Statistics

Page 42, line 10, correction: \( \sqrt{\frac{n(\log n - 1)}{k}} - 1 \).

Chapter 3: Estimation of the Extreme Value Index and Testing

Page 77, line 6, correction: \( \left( \frac{1}{\beta} \right)^{1-\rho} n^{-2\rho/(1-2\rho)} \).
Page 111, line 7, correction: Then (3.6.5) multiplied by \( f(t) \) becomes (…)
line 10, correction: and (3.6.6) multiplied by \( f(t) \) becomes …

Chapter 4: Extreme Quantile and Tail Estimation

Page 128, Fig.4.2(b) label vertical axis, correction: \( \log(1 + \gamma x)/\gamma \)

Page 134, line -12, addendum: Theorem 4.3.1 (de Haan and Rootzén (1993))

Page 138, line 10, addendum: Theorem 4.3.8 (Dijk and de Haan (1992))

Page 148, Sect. 4.6.1. Simulations, correction: (…) \( p_n = 1/(n \log n) = 1.448 \times 10^{-4} \) (…) \( n = 1000 \) (yet \( p_n = 1.086 \times 10^{-4} \) for the Cauchy distribution, which indeed corresponds to 2931.7 in Fig. 4.4)
Table 4.1. Sea level data: 95% asymptotic confidence intervals for quantile.

<table>
<thead>
<tr>
<th></th>
<th>100</th>
<th>200</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moment</td>
<td>(350, 1029.)</td>
<td>(419, 1015.)</td>
<td>(454, 988.)</td>
</tr>
<tr>
<td>PWM</td>
<td>(299, 1017.)</td>
<td>(372, 1049.)</td>
<td>(411, 1018.)</td>
</tr>
</tbody>
</table>

Page 151, Table 4.1, correction:

Table 4.2, clarification: "3rd Quantile" is the same as "0.75 Quantile".

Chapter 6: Basic Theory (in higher dimensional space)

Page 213, Theorem 6.1.9, correction:

Theorem 6.1.9 For any Borel set $A \subset \mathbb{R}_+$ with $\inf_{(x,y) \in A} \max(x, y) > 0$ and any $a > 0$, (...) 

Page 217, Definition 6.1.13, reformulation:

Definition 6.1.13 A distribution function $G$ is called max-stable if there are constants $A_n > 0$, $B_n$, and $D_n$ such that for all $x, y$ and $n = 1, 2, \ldots$, 

$$G^n(A_n x + B_n, C_n y + D_n) = G(x, y).$$

Any distribution function $G$ satisfying (6.1.25) is max-stable (cf. (6.1.28)), and also $G(\alpha x + \beta, \gamma y + \delta)$ where $\alpha > 0$, $\gamma > 0$, $\beta$, and $\delta$ are arbitrary real constants. Since any max-stable distribution is in the class of limit distributions for (6.1.1), we get all the max-stable distributions (i.e. the class of limit distribution functions $G$) this way. The class of limit distribution functions $G_0$ in (6.1.10) is called the class of simple max-stable distributions, “simple” meaning that the marginal distributions are fixed as follows: $G_0(x, \infty) = G_0(\infty, x) = \exp(-1/x)$, $x > 0$.

Page 231, Exercise 6.2, reformulation:

6.2. (..) converges to

$$g(x, y) := 2^{-1} \log(\lambda/(4\pi)) - (4\lambda)^{-1} - \lambda 4^{-1} (x - y)^2 - 2^{-1} (x + y)$$

for $x, y \in \mathbb{R}$. Conclude that

$$\lim_{n \to \infty} n (1 - F_n(a_n x + b_n, a_n y + b_n)) = \iint_{\{s > x\} \cup \{t > y\}} g(s, t) \, ds \, dt,$$

for $x, y \in \mathbb{R}$.

Exercise 6.3 (d), addendum: with $t A^{-1} := \left\{ \left( \frac{x}{y}, \frac{y}{y} \right) : (x, y) \in A \right\}$
Page 232, Exercise 6.6, addendum:

6.6. \( H'(\theta) = r^3q(\theta, r(1 - \theta)) = q(\theta, (1 - \theta)) \) (cf. Coles and Tawn (1991)).

Exercise 6.11, reformulation:

6.11. Let \((V_1, V_2, \ldots, V_d)\) be independent and identically distributed random variables with distribution function \(\exp(-1/x)\), \(x > 0\). Let \(\{r_{i,j}\}_{i=1,2; j=1,\ldots,d}\) be a matrix with positive entries. Show that the random vector \((V_{j=1}^d r_{1,j} V_j, V_{j=1}^d r_{2,j} V_j)\) has a simple max-stable distribution. Find the distribution function. Show that any two dimensional distribution function with Fréchet marginals can be obtained as a limit of elements in this class.

Exercise 6.12, correction: \(\exp - \left( \left( \frac{x}{\alpha} \wedge \frac{y}{\beta} \right)^{-\alpha} + \left( \frac{x}{\gamma} \wedge \frac{y}{\delta} \right)^{-\delta} \right)\)

New Exercise:

6.14. Let \((X, Y)\) have a standard spherically symmetric Cauchy distribution. Show that the probability distribution of \(|X|, |Y|\) is in the domain of attraction of an extreme value distribution with uniform spectral measure \(\Phi\). Show that the probability distribution of \((X, Y)\) is also in a domain of attraction. Find the limit distribution.

Chapter 7: Estimation of the Dependence Structure

Page 268, Exercise 7.3, correction

\[ EW(x_1, \ldots, x_d)W(y_1, \ldots, y_d) = \mu(R(x_1, \ldots, x_d) \cap R(y_1, \ldots, y_d)) \]

Page 269, Exercise 7.4, correction

7.4. (…), and \(N\) indicates a normal probability distribution.

Chapter 9: Basic Theory in \(C[0,1]\)

Page 306, lines 7–19, correction: These lines should be indented, as they belong to part (2) of the proof.

line 11, addendum: First we note that this assumption implies

Page 308, Example 9.4.6, reformulation:

Example 9.4.6 A nice example of a simple max-stable process has already been given by Brown and Resnick (1977). Let \(W^*\) be two-sided Brownian motion:

\[
W^*(s) := \begin{cases} 
W^+(s), & s \geq 0 \\
W^-(s), & s < 0
\end{cases}
\]
where $W^+$ and $W^-$ are independent Brownian motions. For the independent and identically distributed processes $\{V_i\}_{i=1}^{\infty}$ of Corollary 9.4.5 take
\[
\{V_i(s)\}_{s \in \mathbb{R}} := \left\{ e^{W_i(s) - t|s|/2} \right\}_{s \in \mathbb{R}},
\]
where the $W^*_i$ are independent Brownian motions, i.e.,
\[
\{\eta(s)\}_{s \in \mathbb{R}} := \left\{ \sum_{i=1}^{\infty} Z_i e^{W^*_i(s) - t|s|/2} \right\}_{s \in \mathbb{R}},
\]
where $\{Z_i\}_{i=1}^{\infty}$ is a realization of a Poisson point process on $(0, \infty)$ with mean measure $dt/r^2$ and independent of $\{W^*_i\}_{i=1}^{\infty}$. The process $\eta$ is stationary (cf. Section 9.8 below).

Page 311, line 5, addendum: Theorem 9.5.1 (de Haan and Lin (2001))

Page 315, line 3, correction: $(\ldots)$ sup$_{0 \leq t \leq 1} f_s(t) = c$ for all $t \in [0, 1]$.

line 5, addendum: Theorem 9.6.1 (Resnick and Roy (1991) and de Haan (1984))

Section 9.6.2 Stationarity, addendum: An extension of the results of this section to $t \in \mathbb{R}$ is in the Appendix below.

Page 323, Section 9.8 Two Examples, reformulation:

9.8 Two Examples Let us go back to Example 9.4.6 of Section 9.4 (with the aforementioned correction). Let $W^*$ be two-sided Brownian motion,
\[
W^*(s) := \begin{cases} 
W^+(s), & s \geq 0 \\
W^-(s), & s < 0
\end{cases}
\]
where $W^+$ and $W^-$ are independent Brownian motions. Then consider
\[
\{\eta(s)\}_{s \in \mathbb{R}} := \left\{ \sum_{i=1}^{\infty} Z_i e^{W^*_i(s) - t|s|/2} \right\}_{s \in \mathbb{R}}
\]
where $\{Z_i\}_{i=1}^{\infty}$ is a realization of a Poisson point process on $(0, \infty)$ with mean measure $dt/r^2$ and independently, $\{W^*_i\}_{i=1}^{\infty}$ is a sequence of independent two-sided Brownian motions.

Then, to check that the process is stationary just follow the same calculations from pages 323 to 325 with $W$ replaced by $W^*$.

Page 324, line 5, correction:
\[
F_2'(t) = \frac{1}{t \sqrt{s_2}} \phi \left( \frac{\sqrt{s_2}}{2} + \frac{y}{\sqrt{s_2}} + \frac{\log t}{\sqrt{s_2}} \right)
\]

4
Example 9.8.1 Let $Y$ be a random variable with distribution function $1 - 1/x$, $x \geq 1$. Let $W^*$ be two-sided Brownian motion,

$$W^*(s) := \begin{cases} W^+(s), & s \geq 0 \\ W^-(s), & s < 0 \end{cases}$$

where $W^+$ and $W^-$ are independent Brownian motions. Let $Y$ and $W^*$ be independent. Consider the process \(...\) follow the rest of the example with $W$ replaced by $W^*$.

**Example 9.8.2 (Extension of Brown and Resnick (1977))** Let $\{X(s)\}_{s \in \mathbb{R}}$ be an Ornstein-Uhlenbeck process, i.e.,

$$X(s) = 1_{\{s \geq 0\}} e^{-s/2} \left( N + \int_0^s e^{u/2} dW^+(u) \right) + 1_{\{s < 0\}} e^{s/2} \left( N + \int_0^{-s} e^{u/2} dW^-(u) \right)$$

with $N$, $W^+$ and $W^-$ independent, $N$ a standard normal random variable and $W^+$ and $W^-$ standard Brownian motions. Since for $s \neq t$ the random vector $(X(s), X(t))$ is multivariate normal with correlation coefficient less than one, Example 6.2.6 tells us that, relation (9.5.1) cannot hold for any max-stable process in $C[0,1]$; since $Y$ has continuous sample paths, $Y(s)$ and $Y'(t)$ cannot be independent. Hence we compress space in order to create more dependence, i.e., we consider the convergence of

$$\left\{ \sqrt{n} b_n \left( X \left( \frac{s}{b_n^2} \right) - b_n \right) \right\}_{s \in \mathbb{R}}$$

in $C[-s_0, s_0]$ for arbitrary $s_0 > 0$, where $X_1, X_2, \ldots$ are independent and identically distributed copies of $X$ and the $b_n$ are the appropriate normalizing constants for the standard one-dimensional normal distribution, e.g., $b_n = (2 \log n - \log \log n - \log(4\pi))^{1/2}$ (cf. Example 1.1.7). Then

$$b_n \left( X \left( \frac{s}{b_n^2} \right) - b_n \right) = e^{-|s|/2b_n^2} \left( b_n (N - b_n) + b_n \int_0^{\log b_n^2} e^{u/2} dW^\pm(u) + \left( 1 - e^{\log b_n^2} \right) b_n^2 \right)$$

where $W^\pm(s)$ is $W^+(s)$ for $s \geq 0$ and $W^-(s)$ for $s < 0$. Note that uniformly for $|s| \leq s_0$,

$$e^{-|s|/2b_n^2} = 1 + O \left( \frac{1}{b_n^2} \right).$$
Further, since $e^{u/2} = 1 + O \left( \frac{1}{b_n^2} \right)$ for $|u| < s_0/b_n^2$,

$$b_n \int_0^{1/2b_n^2} e^{u/2} dW^\pm(u) = \left( 1 + O \left( \frac{1}{b_n^2} \right) \right) b_n W^\pm \left( \frac{|s|}{b_n^2} \right).$$

Finally, for $|s| \leq s_0$,

$$\left( 1 - e^{1/2|s|} \right) b_n^2 = -\frac{|s|}{2} + O \left( \frac{1}{b_n^2} \right).$$

It follows that

$$b_n \left( X \left( \frac{s}{b_n^2} \right) - b_n \right) = \left( 1 + O \left( \frac{1}{b_n^2} \right) \right) \left( b_n (N - b_n) + b_n W^\pm \left( \frac{|s|}{b_n^2} \right) - \frac{|s|}{2} \right) + O \left( \frac{1}{b_n^2} \right).$$

We write $W^\star (|s|) := b_n W^\pm \left( |s|/b_n^2 \right)$. Then $W^\star$ is also Brownian motion. We have

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} \left( X_i \left( \frac{s}{b_n^2} \right) - b_n \right) \right\}_{s \in \mathbb{R}}
\quad = \left( 1 + O \left( \frac{1}{b_n^2} \right) \right) \left\{ \frac{1}{n} \sum_{i=1}^{n} (b_n (N_i - b_n) + W^\star_i (|s|)) - \frac{|s|}{2} \right\}_{s \in \mathbb{R}} + O \left( \frac{1}{b_n^2} \right).$$

Hence the limit of (2) is the same as that of

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} (b_n (N_i - b_n) + W^\star_i (|s|)) - \frac{|s|}{2} \right\}_{s \in \mathbb{R}}.$$

The rest of the proof runs as in the previous example.

One finds that the sequence of processes (3) converges weakly in $C[-s_0, s_0]$, hence in $C(\mathbb{R})$, to

$$\left\{ \frac{1}{n} \sum_{i=1}^{\infty} (\log Z_i + W^\star_i (|s|)) - \frac{|s|}{2} \right\}_{s \in \mathbb{R}}$$

with $\{Z_i\}$ the point process from (9.8.1).

Note that the point process $\{Z_i\}$ and the random processes $W^\star_i$ are independent.

\textbf{Page 328, Exercise 9.5, correction}  
\textbf{9.5.} (..) $V$ is a continuous stochastic process independent of $Y$ (..)

$$P(\xi(s) > x) = \frac{1}{x} \int_0^x P(V(s) > u) \, du, \quad x > 0.$$  

([..)
Chapter 10: Estimation in C[0, 1]

Page 332, line -2, addendum: Theorem 10.2.1 (de Haan and Lin (2003))

Page 339, line -3, addendum: Theorem 10.4.1 (de Haan and Lin (2003))

Page 341, line -3, correction:

\[
\int (1 - G_{n,S}(s)) x^{\theta(s)-1} \, dx
\]

Page 352, line 6, correction:

\[ \nu_n(S) \stackrel{L^1}{\to} \nu(S). \]

Appendix B: Regular Variation and Extensions

Page 366, line 5, correction: exp \left( \frac{\int_0^t a(v) \, dv}{v} \right)

Page 380, line 9, correction:

\[ (1 - \delta_2) \frac{1 - x^{-\delta_1}}{-\delta_1} \leq \frac{f(tx) - f(t)}{a(t)} \leq (1 + \delta_2) \frac{x^{\delta_1} - 1}{\delta_1} + \delta_2. \]

Page 381, line -8, addendum:

(4) From part (3) of the present proposition it follows (...)

Further and Updated References


Appendix: Extension of Section 9.6.2 Stationarity

**Corollary 9.6.7.A** Let \( \{(Z_i, T_i)\}_{i=1}^{\infty} \) be a realization of a Poisson point process on \([0, \infty) \times \mathbb{R}\) with mean measure \((dr/r^2) \times d\lambda\) (\(\lambda\) Lebesgue measure). If \(\eta\) is a simple max-stable process in \(C^+ (\mathbb{R})\), then there exists a family of functions \(f_s(t) \ (s, t \in \mathbb{R})\) with

1. for each \(t \in \mathbb{R}\) we have a non-negative continuous function \(f_s(t) : \mathbb{R} \to [0, \infty)\),
2. for each \(s \in \mathbb{R}\)
   \[
   \int_{-\infty}^{+\infty} f_s(t) \, dt = 1 ,
   \]  
   (4)
3. for each compact interval \(I \subset \mathbb{R}\)
   \[
   \int_{-\infty}^{+\infty} \sup_{t \in I} f_s(t) \, dt < \infty ,
   \]

such that

\[
\{\eta(s)\}_{s \in \mathbb{R}} = \left\{ \sqrt{\frac{1}{\prod_{i=1}^{d} f_s(T_i)} } \right\}_{s \in \mathbb{R}} .
\]  

(5)

Conversely every process of the form exhibited at the right-hand side of (5) with the stated conditions, is a simple max-stable process in \(C^+ (\mathbb{R})\).

**Proof.** Let \(H\) be a probability distribution function with a density \(H',\) that is positive for all real \(x\). With the functions \(f_s\) from Theorem 9.6.7 define the functions \(\tilde f_s(t) := f_s (H(t)) H'(t)\). Since for any \(s_1, \ldots, s_d \in \mathbb{R}\) and \(x_1, \ldots, x_d \in \mathbb{R}\) positive,

\[
\int_{-\infty}^{+\infty} \max_{1 \leq i \leq d} \tilde f_s(t) x_i \, dt = \int_{-\infty}^{+\infty} \max_{1 \leq i \leq d} f_s(t) x_i \, dt ,
\]

the representation of the corollary follows easily from that of Theorem 9.6.7.

**Definition 9.6.9.A** A mapping \(\Phi\) from \(L_1^+\) (the non-negative integrable functions on \(\mathbb{R}\)) to \(L_1^+\) is called a piston if for every \(h \in L_1^+\)

\[
\Phi (h(t)) = r(t) h (H(t))
\]

with \(H\) a one-to-one measurable mapping from \(\mathbb{R}\) to \(\mathbb{R}\) and \(r\) a positive measurable function, such that for every \(h \in L_1^+\)

\[
\int_{-\infty}^{+\infty} \Phi (h(t)) \, dt = \int_{-\infty}^{+\infty} h(t) \, dt .
\]

8
**Theorem 9.6.10.A** Let \( \{(Z_i, T_i)\}_{i=1}^{\infty} \) be a realization of a Poisson process on \((0, \infty) \times \mathbb{R}\) with mean measure \((dr/r^2) \times d\lambda\) (Lebesgue measure).

If the stochastic process \( \{\eta(s)\}_{s \in \mathbb{R}} \) is simple max-stable, strictly stationary and continuous a.s., then there is a function \( h \) in \( L_1^+ \) with \( \int_{-\infty}^{\infty} h(t) \, dt = 1 \) and a continuous group of pistons \( \{\Phi_s\}_{s \in \mathbb{R}} \) (continuous i.e., \( \Phi_{s_n}(h(t)) \to \Phi_s(h(t)) \) as \( s_n \to s \) for almost all \( t \in \mathbb{R} \)) with

\[
\int_{-\infty}^{\infty} \sup_{t \in I} \Phi_s(h(t)) \, dt < \infty
\]

for each compact interval \( I \), such that

\[
\{\eta(s)\}_{s \in \mathbb{R}} \overset{d}{=} \left\{ \sum_{i=1}^{\infty} Z_i \Phi_s(h(T_i)) \right\}_{s \in \mathbb{R}}.
\]

(6)

Conversely every stochastic process of the form exhibited at the right-hand side of (6) with the stated conditions, is simple max-stable, strictly stationary and a.s. continuous.

**Proof.** Just replace everywhere in the proof of Theorem 9.6.10, \( t \in [0, 1] \) by \( t \in \mathbb{R} \).