# A representation of max-stable processes *

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**Abstract**

A standardized max-stable process on $C(S)$ with $S$ a compact subset of a Euclidean space has a simple representation involving a more or less arbitrary stochastic process.

Keywords: max-stable process, Poisson point process, Ornstein-Uhlenbeck process

## 1 Introduction

Max-stable processes form a useful tool to model extremal behaviour in e.g. environmental processes like rainfall.

Probably the earliest example of a max-stable process has been given in Brown and Resnick (1977): Let $O_1, O_2, \cdots$ be independent and identically distributed (i.i.d.) copies of the Ornstein-Uhlenbeck process. Consider the sequence of processes

$$\{M_n(s)\}_{s\in\mathbb{R}} := \left\{ \bigvee_{i=1}^{n} b_n \left( O_i \left( s/b_n^2 \right) - b_n \right) \right\}_{s\in\mathbb{R}}$$

where the maximum is taken pointwise and $b_n = (2\log n - \log\log n - \log 4\pi)^{1/2}$. Note that the $b_n$ are chosen such that the $M_n(0)$ has a non-degenerate limit distribution. Then the sequence $\{M_n(s)\}_{s\in\mathbb{R}}$ converges in distribution in $C(\mathbb{R})$ to a process $\{\eta(s)\}_{s\in\mathbb{R}}$ with the following structure. $C(\mathbb{R})$ denotes the space of continuous functions on $\mathbb{R}$.

Consider a Poisson point process on $\mathbb{R}_+$ with mean measure $x^{-2}dx$. Let $\{X_i\}_{i=1}^{\infty}$ be an enumeration of the points of a realization of the point process. Also consider an i.i.d. sequence of Brownian motions $\{W_i(s)\}_{s\in\mathbb{R}}$, independent

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of the point process, where $W_i(0) = 0$ for all $i = 1, 2, \ldots$ and the process goes off in a symmetric way in both directions. Then

$$\{\eta(s)\}_{s\in\mathbb{R}} \stackrel{d}{=} \left\{ \bigvee_{i=1}^{\infty} X_i e^{W_i(s)-|s|/2} \right\}_{s\in\mathbb{R}} .$$

We shall show that this structure - points of a Poisson point process marked by i.i.d. stochastic processes - is valid for any simple max-stable process in $C(S)$ with $S$ a compact subset of a Euclidean space. $C(S)$ is the space of continuous functions $f$ on $S$ equipped with the supremum norm $|f|_\infty = \sup_{s\in S} |f(s)|$.

# 2 Main result

Let $\xi$ be a stochastic process on $C(S)$ with non-degenerate marginals, i.e. $\xi(s)$ is non-degenerate for all $s \in S$. The process $\xi$ is *max-stable* if there exist continuous functions $a_n > 0$ and $b_n$, defined on $S$, such that if $\xi_1, \xi_2, \cdots, \xi_n$ are i.i.d. copies of $\xi$,

$$\left\{ \bigvee_{i=1}^{n} \frac{\xi_i(s) - b_n(s)}{a_n(s)} \right\}_{s\in S} \stackrel{d}{=} \{\xi(s)\}_{s\in S} .$$

The probabilistic structure of those processes is fully captured, through a transformation of the marginal distributions, by a corresponding *simple* max-stable process. A stochastic process $\eta$ in $C(S)$ with non-degenerate marginals is simple max-stable if, with $\eta_1, \eta_2, \cdots, \eta_n$ i.i.d. copies of $\eta$,

$$\left\{ \frac{1}{n} \bigvee_{i=1}^{n} \eta_i(s) \right\}_{s\in S} \stackrel{d}{=} \{\eta(s)\}_{s\in S}$$

and $P\{\eta(s) \leq 1\} = e^{-1}$ for $s \in S$. The first requirement determines the type of the marginal distribution, the latter fixes the scale (cf. Giné, Hahn and Vatan (1990)). We shall prove the following result.

**Theorem 2.1.** *Let the process $\eta$ be simple max-stable in $C(S)$. Consider a Poisson point process on $(0, \infty)$ with mean measure $dr/r^2$. Let $\{Z_i\}_{i=1}^{\infty}$ be the enumeration of the points of a realization of the point process. We can find i.i.d. non-negative stochastic processes $V, V_1, V_2, \cdots$ in $C(S)$ with*

$$\begin{aligned} EV(s) &= 1, \quad \textit{for } s \in S, \\ E\sup_{s\in S} V(s) &< \infty \end{aligned} \tag{1}$$

*such that*

$$\eta \stackrel{d}{=} \bigvee_{i=1}^{\infty} Z_i V_i .$$

*Conversely each process with this representation is simple max-stable.*

*The process $V$ can be chosen is such a way that*

$$\sup_{s\in S} V(s) = c \quad a.s. \tag{2}$$

*with c some positive constant.*

For the proof we use the following result by Giné, Hahn and Vatan (1990).

**Proposition 2.1.** *Let $\eta$ be a simple max-stable process in $C(S)$. There exists a finite Borel measure $\rho$ on $C_1^+ := \{f \in C(S) : f \geq 0, |f|_\infty = 1\}$ with the property that*

$$\int_{C_1^+} f(s) d\rho(f) = 1 \text{ for all } s \in S \ ,$$

*such that*

$$\eta \stackrel{d}{=} \bigvee_{i=1}^{\infty} \eta_i$$

*where $\{\eta_i(s)\}_{s\in S} := \{Z_i \pi_i(s)\}_{s\in S}$ and $(Z_i, \pi_i)$, $i = 1, 2, \ldots$, are the points of a Poisson point process on $(0,\infty) \times C_1^+$ with mean measure $r^{-2}dr \times d\rho$.*

*Conversely each process with this representation is simple max-stable.*

We also need the following auxiliary result.

**Lemma 2.1.** *Suppose $P$ is a Poisson point process on the product space $S_1 \times S_2$ with $S_1$ and $S_2$ metric spaces and the intensity measure is $\nu = \nu_1 \times \nu_2$ where $\nu_1$ is not bounded and $\nu_2$ is a probability measure. The process can be generated in the following way: let $\{U_i\}$ be an enumeration of the points of the Poisson point process on $S_1$ with intensity measure $\nu_1$ and let $V_1, V_2, \ldots$ be independent and identically distributed random elements of $S_2$ with probability distribution $\nu_2$. Then the counting measure $N$ defined by*

$$N(A_1 \times A_2) := \sum_{i=1}^{\infty} 1_{\{(U_i,V_i)\in A_1\times A_2\}}$$

*for Borel sets $A_1 \subset S_1$, $A_2 \subset S_2$, has the same distribution as the point process $P$.*

*Proof.* We need to prove that the number of points of the set $\{(U_i, V_i)\}_{i=1}^{\infty}$ in two disjoint Borel sets are independent (which is trivial) and that the number of points $N(A_1 \times A_2)$ in a Borel set $A_1 \times A_2$, with $A_1 \subset S_1$ and $A_2 \subset S_2$, has a

Poisson distribution with mean measure $\nu_1(A_1)\nu_2(A_2)$. Now

$$\begin{aligned}
&P\left(N(A_1 \times A_2) = k\right) \\
&= \sum_{r=k}^{\infty} P\left(N(A_1 \times A_2) = k \mid \text{the number of points in } A_1 = r\right) \\
&\quad \frac{\left(\nu_1\left(A_1\right)\right)^r}{r!} e^{-\nu_1(A_1)} \\
&= \sum_{r=k}^{\infty} \frac{r!}{(r-k)!k!} \left(\nu_2\left(A_2\right)\right)^k \left(1-\nu_2\left(A_2\right)\right)^{r-k} \frac{\left(\nu_1\left(A_1\right)\right)^r}{r!} e^{-\nu_1(A_1)} \\
&= \frac{\left(\nu_1\left(A_1\right)\nu_2\left(A_2\right)\right)^k}{k!} e^{-\nu_1(A_1)} \sum_{r=k}^{\infty} \frac{\left(1-\nu_2\left(A_2\right)\right)^{r-k}}{(r-k)!} \left(\nu_1\left(A_1\right)\right)^{r-k} \\
&= \frac{\left(\nu_1\left(A_1\right)\nu_2\left(A_2\right)\right)^k}{k!} e^{-\nu_1(A_1)\nu_2(A_2)} \, .
\end{aligned}$$

□

*Proof of Theorem 2.1.* Let $\eta$ be simple max-stable and $\{(Z_i, \pi_i)\}$ the points of the point process of Proposition 2.1. Let $\{(\tilde{Z}_i, \tilde{\pi}_i)\}$ be the points of a Poisson point process on $(0,\infty) \times C_1^+$ with mean measure

$$\rho\left(C_1^+\right) \frac{dr}{r^2} \times \frac{d\rho}{\rho\left(C_1^+\right)} \, .$$

Then the product measure remains the same and hence

$$\bigvee_{i=1}^{\infty} \tilde{Z}_i \tilde{\pi}_i \stackrel{d}{=} \bigvee_{i=1}^{\infty} Z_i \pi_i \stackrel{d}{=} \eta \, .$$

Next consider the collection of points $\{(\tilde{\tilde{Z}}_i, \tilde{\tilde{\pi}}_i)\}$ defined for $i = 1, 2, \ldots$ by

$$\begin{aligned}
\tilde{\tilde{Z}}_i &:= \tilde{Z}_i / \rho\left(C_1^+\right) \\
\tilde{\tilde{\pi}}_i &:= \tilde{\pi}_i \, \rho\left(C_1^+\right) \, .
\end{aligned}$$

Then $\{(\tilde{\tilde{Z}}_i, \tilde{\tilde{\pi}}_i)\}$ represents a Poisson point process on $(0,\infty) \times C_\rho^+$ with

$$C_\rho^+ := \{f \in C(S) : f \geq 0, |f|_\infty = \rho(C_1^+)\} \, .$$

We now argue that its intensity measure is

$$r^{-2} dr \times dQ$$

with $Q$ a probability measure. The intensity measure of the first component is

$$\int_{\left\{z : z/\rho\left(C_1^+\right) \in A\right\}} \rho\left(C_1^+\right) \frac{dz}{z^2} = \int_A \frac{dz}{z^2}$$

for a Borel set $A$ of $(0,\infty)$ . The intensity measure of the second component is

$$Q(\cdot) := \frac{\rho\left\{f : f \geq 0, |f|_\infty = 1\ , f\rho\left(C_1^+\right) \in \cdot\right\}}{\rho\left(C_1^+\right)}$$

which is a probability measure.

Application of the Lemma now provides the representation of the theorem with $V$ satisfying (2).

For the converse just follow the steps backwards.

It remains to prove that for the converse the requirement $\sup_{s\in S} V(s) = c$ a.s. can be relaxed to $E\sup_{s\in S} V(s) < \infty$. Checking the proof of the Proposition in Giné, Hahn and Vatan (1990), one sees that their condition which in our notation is $\sup_{s\in S} \tilde{\tilde{\pi}}_i(s) = \rho(C_1^+)$ for all $i$, is used only to ensure the finiteness of the process $\eta$. But this also follows from our weaker assumption: we consider now a probability measure $Q$ on the space

$$C^* := \{f \in C(S) : f \geq 0, |f|_\infty > 0\}$$

with the property

$$\int_{C^*} |f|_\infty dQ(f) < \infty\ .$$

Then

$$\begin{aligned} P&\left\{\sup_{s\in S} \eta(s) \leq x\right\} \\ &= P\left\{\text{none of the points } \tilde{\tilde{Z}}_i\tilde{\tilde{\pi}}_i, i = 1,2,\ldots, \text{ is in the set } \{f : f \geq 0, |f|_\infty > x\}\right\} \\ &= \exp\left(-\iint\limits_{|f|_\infty > x} \frac{dr}{r^2} dQ(f)\right) = \exp\left(-\frac{1}{x}\int_{C^*} |f|_\infty\, dQ(f)\right) > 0\ . \end{aligned}$$

Hence the process $\eta$ is bounded. □

**Remark 2.1.** *The result can also be expressed as a representation using a family of (deterministic) spectral functions acting on a homogeneous Poisson process on* $(0,\infty)\times[0,1]$ *as in de Haan (1984): there exist non-negative measurable functions* $f_s(u)$ *where* $s \in S$, $u \in [0,1]$ *with*

*i. for each* $u \in [0,1]$ *the function* $f_s(u)$ *is continuous in* $s$,

*ii. for each* $s \in S$

$$\int_0^1 f_s(u)du = 1\ ,$$

*iii.*

$$\int_0^1 \sup_{s\in S} f_s(u)du < \infty\ ,$$

*such that*

$$\{\eta(s)\}_{s\in S} \stackrel{d}{=} \left\{ \bigvee_{i=1}^{\infty} X_i f_s(Y_i) \right\}_{s\in S}$$

*where* $\{(X_i, Y_i)\}_{i=1}^{\infty}$ *is an enumeration of the points of a Poisson point process on* $(0,\infty)\times[0,1]$ *with mean measure* $dr/r^2 \times d\lambda$ *(*$\lambda$ *is Lebesgue measure).*

**Corollary 2.1 (Sample behaviour).** *Consider the representation of Theorem 2.1 using the Poisson point process represented by* $\{(Z_i, V_i)\}_{i=1}^{\infty}$. *With probability one there exists a finite collection* $\{(Z_j, V_j)\}_{j=1}^{k}$ *such that*

$$\eta \stackrel{d}{=} \bigvee_{j=1}^{k} Z_j V_j \; .$$

*Proof.* By Corollary 3.4 , Giné, Hahn and Vatan (1990),

$$P\{\eta > 0\} = 1 \; .$$

Hence on a set of probability one we have both that $\eta > 0$ and that for all $\epsilon > 0$ there are only finitely many $Z_i$ with $Z_i > \epsilon$. The result follows. □

**Remark 2.2.** *Consider the stochastic process on a compact subset* $S$ *of a Euclidean space, defined by* $\zeta := YV$ *where* $Y$ *is a random variable with distribution function* $1 - 1/x$, $x \geq 1$ *and* $V$ *a continuous stochastic process satisfying* (1). *Let* $\zeta_1, \zeta_2, \ldots$ *be i.i.d. copies of* $\zeta$. *It is easy to see that* $n^{-1} \vee_{i=1}^{n} \zeta_i$ *converges in* $C(S)$ *to a simple max-stable process that has the representation of Theorem 2.1 with the same auxiliary process* $V$. *Moreover for* $a > 1$

$$\left\{a^{-1}\zeta(s) | \zeta(0) > a\right\}_{s\in S}$$

*has the same distribution as* $\zeta$. *This property resembles a corresponding property for generalized Pareto distributions in finite-dimensional space. Note that the marginal distributions of this process are not Pareto: for* $x > 0$

$$P\{\zeta(s) > x\} = x^{-1} \int_0^x P\{V(s) > u\} du \; .$$

# References


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