On the estimation of the probability of a failure set

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Abstract

The theory of estimating the probability of a failure set (i.e. a set beyond the range of the available observations) is well-known (see e.g. Joe, Smith and Weissman (1992); Ledford and Tawn (1997); de Haan and Sinha (1999); Draisma, Drees, Ferreira and de Haan (2004)). In the literature the conditions imposed on the probability distribution and on the failure set are very complicated and not transparent. We present a new treatment that simplifies conditions and proofs (e.g. no more Vapnik-Cervonenkis classes). For simplicity in finite-dimensional space we consider the two-dimensional case. Generalization to higher dimensions is immediate. The method allows us to prove a similar result in functions space which is completely new.

Keywords and phrases: multivariate extreme value distribution; failure probability; estimation; infinite dimensional

1 Introduction and result in Euclidean space

Let (X, Y), $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be i.i.d. random variables with distribution function F. Consider a failure set C which is outside the range of the observations in the north-eastern corner. One wants to estimate the probability that a future observation falls in the set C.

In order to formalize the situation we assume that the set C in fact depends on n ($C = C_n$) and that $P(C_n) = O(1/n)$, as $n \to \infty$.

Moreover we assume that there exists $(v_n, w_n) \in \partial C_n$ such that $(-\infty, v_n] \times (-\infty, w_n] \cap C_n = \emptyset$. Further one assumes that the distribution function F is in the domain of attraction of an extreme value distribution. Next we introduce an estimator $\hat{P}(C_n)$. The aim is to prove that $\hat{P}(C_n)/P(C_n) \xrightarrow{P} 1$

as $n \to \infty$. This has been achieved in Draisma, Drees, Ferreira and de Haan (2004), albeit under complicated conditions. We present a new approach and set out our conditions first.

1. F is in the domain of attraction of an extreme value distribution i.e. for functions $a_1, a_2 > 0$, b_1, b_2 and real parameters γ_1 and γ_2

$$\lim_{t\to\infty} tP\left\{\left(\left(1+\gamma_1\frac{X-b_1(t)}{a_1(t)}\right)^{1/\gamma_1}, \left(1+\gamma_2\frac{Y-b_2(t)}{a_2(t)}\right)^{1/\gamma_2}\right)\in A\right\} = \nu(A) \tag{1}$$

   for each Borel set $A \subset [0,\infty)^2$ with $\inf_{(x,y)\in A}\max(x,y) > 0$ and $\nu(\partial A) = 0$ where ν is a measure on $[0,\infty)^2 \setminus \{(0,0)\}$ which is homogeneous i.e. for $a > 0$

$$\nu(aA) = a^{-1}\nu(A) \tag{2}$$

   where aA is the set obtained by multiplying all elements of A by a.

2. We need estimators $\hat{\gamma}_1, \hat{\gamma}_2, \hat{a}_1\left(\frac{n}{k}\right), \hat{a}_2\left(\frac{n}{k}\right), \hat{b}_1\left(\frac{n}{k}\right), \hat{b}_2\left(\frac{n}{k}\right)$ such that for some sequence $k = k(n) \to \infty, k/n \to 0, n \to \infty$,

$$\sqrt{k}\left(\hat{\gamma}_i - \gamma_i, \frac{\hat{a}_i\left(\frac{n}{k}\right)}{a_i\left(\frac{n}{k}\right)} - 1, \frac{\hat{b}_i\left(\frac{n}{k}\right) - b_i\left(\frac{n}{k}\right)}{a_i\left(\frac{n}{k}\right)}\right) = (O_P(1), O_P(1), O_P(1))$$

   for $i = 1, 2$. There are several known estimators with this behaviour under suitable second order extreme value conditions on F.

3. C_n is open and there exists $(v_n, w_n) \in \partial C_n$ such that $(-\infty, v_n] \times (-\infty, w_n] \cap C_n = \emptyset$.

4. The set

$$S := \left\{\left(\frac{1}{c_n}\left(1+\gamma_1\frac{X-b_1\left(\frac{n}{k}\right)}{a_1\left(\frac{n}{k}\right)}\right)^{1/\gamma_1}, \frac{1}{c_n}\left(1+\gamma_2\frac{Y-b_2\left(\frac{n}{k}\right)}{a_2\left(\frac{n}{k}\right)}\right)^{1/\gamma_2}\right) : (X,Y)\in C_n\right\} \tag{3}$$

in $\mathbb{R}_+^2$ does not depend on n where

$$
\begin{aligned}
c_n &:= \sqrt{q_n^2 + r_n^2} \\
q_n &:= \left(1 + \gamma_1 \frac{v_n - b_1\left(\frac{n}{k}\right)}{a_1\left(\frac{n}{k}\right)}\right)^{1/\gamma_1} \\
r_n &:= \left(1 + \gamma_2 \frac{w_n - b_2\left(\frac{n}{k}\right)}{a_2\left(\frac{n}{k}\right)}\right)^{1/\gamma_2} .
\end{aligned}
\tag{4}
$$

Further: S has positive distance from the origin and $\nu(\partial S) = 0$. Note that (3) implies that $(q_n/c_n, r_n/c_n) \in \partial S$. Hence q_n/r_n does not depend on n. We suppose $0 < q_n/r_n < \infty$. We require that $c_n \to \infty$, $n \to \infty$. Moreover (cf. Condition 1) as $n \to \infty$

$$
\begin{aligned}
&P(C_n) = \\
&P \left\{ \left(1 + \gamma_1 \frac{X - b_1\left(\frac{n}{k}\right)}{a_1\left(\frac{n}{k}\right)}\right)^{1/\gamma_1}, \left(1 + \gamma_2 \frac{Y - b_2\left(\frac{n}{k}\right)}{a_2\left(\frac{n}{k}\right)}\right)^{1/\gamma_2} \in c_n S \right\} \\
&\sim \frac{k}{n}\nu(c_n S) = \frac{k}{n c_n}\nu(S) .
\end{aligned}
$$

This can be guaranteed by a proper second order condition on the probability distribution.

5. $\gamma_1, \gamma_2 > -1/2$ and

$$
\lim_{n \to \infty} \frac{w_{\gamma_1 \wedge \gamma_2}(c_n)}{\sqrt{k}} = 0
$$

where

$$
w_\gamma(x) := x^{-\gamma} \int_1^x s^{\gamma - 1} \log s \, ds .
$$

Write $p_n := P(C_n)$. Our estimator $\hat{p}_n$ for p_n is more or less the traditional one.

Define (with the convention $0^{1/\hat{\gamma}_i} = 0$ regardless the sign of $\hat{\gamma}_i$)

$$
\begin{aligned}
\hat{q}_n &:= \left(1 + \hat{\gamma}_1 \frac{v_n - \hat{b}_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}_1} \\
\hat{r}_n &:= \left(1 + \hat{\gamma}_2 \frac{w_n - \hat{b}_2\left(\frac{n}{k}\right)}{\hat{a}_2\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}_2} \\
\hat{c}_n &:= \sqrt{\hat{q}_n^2 + \hat{r}_n^2}
\end{aligned}
\tag{5}
$$

and

$$\hat{S}_n := \left\{ \left(\frac{1}{\hat{c}_n} \left(1 + \hat{\gamma}_1 \frac{x + \hat{b}_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)} \right)^{1/\hat{\gamma}_1}, \frac{1}{\hat{c}_n} \left(1 + \hat{\gamma}_2 \frac{y - \hat{b}_2\left(\frac{n}{k}\right)}{\hat{a}_2\left(\frac{n}{k}\right)} \right)^{1/\hat{\gamma}_2} \right) : (x,y) \in C_n \right\}. \tag{6}$$

Then we define

$$\hat{p}_n := \frac{1}{n\hat{c}_n} \sum_{i=1}^{n} 1_{\left\{ \left(\left(1 + \hat{\gamma}_1 \frac{X_i + \hat{b}_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)} \right)^{1/\hat{\gamma}_1}, \left(1 + \hat{\gamma}_2 \frac{Y_i - \hat{b}_2\left(\frac{n}{k}\right)}{\hat{a}_2\left(\frac{n}{k}\right)} \right)^{1/\hat{\gamma}_2} \right) \in \hat{S}_n \right\}}. \tag{7}$$

We shall prove

Theorem 1.1. *Under Conditions 1-5 with $\hat{p}_n$ as in (7)*

$$\frac{\hat{p}_n}{p_n} \xrightarrow{P} 1$$

as $n \to \infty$, provided $\nu(S) > 0$.

Remark 1.1. *The condition $\nu(S) > 0$ can be violated under asymptotic independence in (1). In that case a similar statement with similar proof applies under the Ledford and Tawn (1997) condition. We omit the details.*

Remark 1.2. *Note that $\hat{q}_n, \hat{r}_n, \hat{S}_n, \hat{p}_n$ may not be defined if $1 + \hat{\gamma}_1(X_i - \hat{b}_1(n/k))/\hat{a}_1(n/k) \leq 0$ for some X_i and similarly with the second component. However when checking the proofs one sees that when $n \to \infty$, the probability that this happens tends to zero.*

2 Proof of Theorem 1.1

The proof of the Theorem will follow from three analytical lemmas and four propositions.

Lemma 2.1. *Let $f_n(x)$ and $g_n(x)$ be strictly increasing continuous functions for all n, $\lim_{n\to\infty} f_n(x) = x$ and $\lim_{n\to\infty} g_n(x) = x$ for $x > 0$. For an open set O let*

$$O_n := \{f_n(x), g_n(y) : (x,y) \in O\}.$$

Then

$$1_{O_n}(x,y) := 1_{\{(x,y)\in O_n\}} \to 1_O(x,y) := 1_{\{(x,y)\in O\}}$$

for $(x,y)\in O$.

Proof. Take $(x,y)\in O$ and $\varepsilon>0$ such that $(x-\varepsilon, y-\varepsilon)\in O$. For $n\ge n_0$ we have $f_n^{\leftarrow}(x) > x-\varepsilon$ and $g_n^{\leftarrow}(y) > y-\varepsilon$. Hence $(f_n^{\leftarrow}(x), g_n^{\leftarrow}(y))\in O$ for $n\ge n_0$. It follows that

$$1_O\left(f_n^{\leftarrow}(x), g_n^{\leftarrow}(y)\right) \to 1$$

for $(x,y)\in O$. Now

$$(x,y)\in O_n \Leftrightarrow (f_n^{\leftarrow}(x), g_n^{\leftarrow}(y))\in O \Leftrightarrow 1_O\left(f_n^{\leftarrow}(x), g_n^{\leftarrow}(y)\right) = 1.$$

Hence the conclusion. □

Lemma 2.2. *For all real* γ *and* $x>0$

$$\lim_{n\to\infty}\left(1+\gamma\left\{(1+o_1(1))\,\frac{x^{\gamma+o_2(1)}-1}{\gamma+o_2(1)}+o_3(1)\right\}\right)^{1/\gamma} = x$$

where the three o_i*-terms are not necessarily the same.*

Lemma 2.3. *For all real* γ *and* $x>0$, *with* $\gamma_n\to\gamma$ *and* $c_n\to\infty (n\to\infty)$,

$$\lim_{n\to\infty}\frac{1}{c_n}\left(1+\gamma_n\left\{(1+O_1(\gamma_n-\gamma))\,\frac{(c_n x)^{\gamma}-1}{\gamma}+O_2(\gamma_n-\gamma)\right\}\right)^{1/\gamma_n} = x$$

provided

$$\lim_{n\to\infty}(\gamma_n-\gamma)c_n^{-\gamma}\int_1^{c_n} s^{\gamma-1}\log s\; ds = 0\,. \tag{8}$$

Proof. For $\gamma\neq 0$ the left hand side can be written as

$$\frac{1}{c_n}\left[(c_n x)^{\gamma}+(c_n x)^{\gamma}\,O_1(\gamma_n-\gamma)+O_2(\gamma_n-\gamma)\right]^{1/\gamma_n}$$
$$= \frac{(c_n x)^{\gamma/\gamma_n}}{c_n}\left[1+O_1(\gamma_n-\gamma)+(c_n x)^{-\gamma}\,O_2(\gamma_n-\gamma)\right]^{1/\gamma_n}.$$

We deal with the two factors separately. Note that (8) implies

$$\begin{cases} (\gamma_n-\gamma)\log c_n \to 0 & ,\text{if } \gamma>0 \\ (\gamma_n-\gamma)c_n^{-\gamma}\to 0 & ,\text{if } \gamma<0\,, \end{cases}$$

hence whatever the value of $\gamma \neq 0$

$$(\gamma_n - \gamma)c_n^{-\gamma} \to 0 \quad \text{and} \quad (\gamma_n - \gamma)\log c_n \to 0, \quad n \to \infty \,.$$

The result follows for $\gamma \neq 0$.

Next consider $\gamma = 0$. We prove that the inverse function of the left hand side converges to the identity:

$$\begin{aligned}
&\frac{1}{c_n}\exp\left[(1+O_1(\gamma_n-\gamma))\,\frac{(c_n x)^{\gamma_n}-1}{\gamma_n} + O_2(\gamma_n-\gamma)\right] \\
&\quad = \; x\exp\left[(1+O_1(\gamma_n-\gamma))\left(\frac{(c_n x)^{\gamma_n}-1}{\gamma_n} - \log c_n x\right)\right. \\
&\qquad \left. + O_1(\gamma_n-\gamma)\log(c_n x) + O_2(\gamma_n-\gamma)\right] .
\end{aligned}$$

Now note that

$$\begin{aligned}
&\left|\frac{(c_n x)^{\gamma_n}-1}{\gamma_n} - \log(c_n x)\right| \\
&\qquad = \left|\gamma_n \int_1^{c_n x}\int_1^{s} u^{\gamma_n}\frac{du}{u}\frac{ds}{s}\right| \leq |\gamma_n|\,(c_n x)^{|\gamma_n|}\frac{1}{2}\log^2(c_n x)
\end{aligned}$$

The proof is completed by noting that (8) implies $\gamma_n \log^2 c_n \to 0, n \to \infty$, if $\gamma = 0$. □

Next we introduce some transformations.

$$\begin{aligned}
R_n(x,y) &:= \left(\left(1+\gamma_1\frac{x-b_1\left(\frac{n}{k}\right)}{a_1\left(\frac{n}{k}\right)}\right)^{1/\gamma_1}, \left(1+\gamma_2\frac{y-b_2\left(\frac{n}{k}\right)}{a_2\left(\frac{n}{k}\right)}\right)^{1/\gamma_2}\right) \\
\hat{R}_n(x,y) &:= \left(\left(1+\hat{\gamma}_1\frac{x-\hat{b}_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}_1}, \left(1+\hat{\gamma}_2\frac{y-\hat{b}_2\left(\frac{n}{k}\right)}{\hat{a}_2\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}_2}\right) \\
Q_n(x,y) &:= \frac{1}{c_n}R_n(x,y) \\
\hat{Q}_n(x,y) &:= \frac{1}{\hat{c}_n}\hat{R}_n(x,y) \,.
\end{aligned}$$

Proposition 2.1. *Let S be an open Borel set in $\mathbb{R}_+^2$ with $\nu(S) > 0, \nu(\partial S) = 0$ and with positive distance from the origin. Suppose Condition 1 holds. Define*

$$\begin{aligned} \tilde{\nu}_n(S) &:= \frac{1}{k}\sum_{i=1}^{n} 1_{\left\{\left(\left(1+\gamma_1 \frac{X_i - b_1\left(\frac{n}{k}\right)}{a_1\left(\frac{n}{k}\right)}\right)^{1/\gamma_1}, \left(1+\gamma_2 \frac{Y_i - b_2\left(\frac{n}{k}\right)}{a_2\left(\frac{n}{k}\right)}\right)^{1/\gamma_2}\right) \in S\right\}} \\ &= \frac{1}{k}\sum_{i=1}^{n} 1_{\{R_n(X_i,Y_i)\in S\}} \ . \end{aligned}$$

Then with k satisfying $k = k(n) \to \infty$, $k/n \to 0$, as $n \to \infty$,

$$\tilde{\nu}_n(S) \xrightarrow{P} \nu(S) \ .$$

Proof. By Condition 1 we find easily

$$\lim_{n\to\infty} E e^{it\,\tilde{\nu}_n(S)} = e^{it\,\nu(S)}$$

for all t. ☐

Proposition 2.2. *Let S be as in Proposition 2.1. Suppose that Condition 1 holds and there are estimators $\left(\hat{\gamma}_i, \hat{a}_i(n/k), \hat{b}_i(n/k)\right)$ such that*

$$\left(\hat{\gamma}_i - \gamma, \frac{\hat{a}_i\left(\frac{n}{k}\right)}{a_i\left(\frac{n}{k}\right)} - 1, \frac{\hat{b}_i\left(\frac{n}{k}\right) - b_i\left(\frac{n}{k}\right)}{a_i\left(\frac{n}{k}\right)}\right) \xrightarrow{P} (0,0,0) \tag{9}$$

for $i = 1, 2$ as $n \to \infty$ (the latter follows from Condition 2). Define

$$\begin{aligned} \hat{\nu}_n(S) &:= \frac{1}{k}\sum_{i=1}^{n} 1_{\left\{\left(\left(1+\hat{\gamma}_1 \frac{X_i - \hat{b}_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}_1}, \left(1+\hat{\gamma}_2 \frac{Y_i - \hat{b}_2\left(\frac{n}{k}\right)}{\hat{a}_2\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}_2}\right) \in S\right\}} \\ &= \frac{1}{k}\sum_{i=1}^{n} 1_{\left\{\hat{R}_n(X,Y)\in S\right\}} \ . \end{aligned}$$

Then as $n \to \infty$

$$\hat{\nu}_n(S) \xrightarrow{P} \nu(S) \ .$$

Proof. First invoke a Skorohod construction so that we may pretend that the left hand side of (9) converges to zero almost surely and that $\tilde{\nu}_n(S) \to \nu(S)$ almost surely. Note that

$$\begin{aligned} \hat{\nu}_n(S) &= \frac{1}{k}\sum_{i=1}^{n} 1_{\{(X,Y)\in \hat{R}_n^{\leftarrow}(S)\}} \\ &= \frac{1}{k}\sum_{i=1}^{n} 1_{\{R_n(X,Y)\in R_n\ \hat{R}_n^{\leftarrow}(S)\}} \\ &= \tilde{\nu}(S_n) \end{aligned}$$

with

$$S_n := R_n\ \hat{R}_n^{\leftarrow}(S) = \left\{R_n\ \hat{R}_n^{\leftarrow}(x,y) : (x,y) \in S\right\} .$$

Now

$$\begin{aligned} R_n\ \hat{R}_n^{\leftarrow}(x,y) \quad = \quad & \left(\left(1+\gamma_1\left[\frac{\hat{a}_1\left(\frac{n}{k}\right)}{a_1\left(\frac{n}{k}\right)}\frac{x^{\hat{\gamma}_1}-1}{\hat{\gamma}_1}+\frac{\hat{b}_1\left(\frac{n}{k}\right)-b_1\left(\frac{n}{k}\right)}{a_1\left(\frac{n}{k}\right)}\right]\right)^{1/\gamma_1}\right. , \\ & , \quad \left.\left(1+\gamma_2\left[\frac{\hat{a}_2\left(\frac{n}{k}\right)}{a_2\left(\frac{n}{k}\right)}\frac{y^{\hat{\gamma}_2}-1}{\hat{\gamma}_2}-\frac{\hat{b}_2\left(\frac{n}{k}\right)-b_2\left(\frac{n}{k}\right)}{a_2\left(\frac{n}{k}\right)}\right]\right)^{1/\gamma_2}\right) , \end{aligned}$$

hence we have by Lemma 2.2

$$R_n\ \hat{R}_n^{\leftarrow}(x,y) \to (x,y) \tag{10}$$

for $x, y > 0$ a.s. and hence by Lemma 2.1

$$1_{R_n\ \hat{R}_n^{\leftarrow}(S)}(x,y) \to 1_S(x,y) \tag{11}$$

for all $(x,y) \in S$ a.s.

Further since S has positive distance to the origin, there are $s > 0$ such that

$$S \subset \{(x,y) : x > s \quad \text{or} \quad y > s\} =: D .$$

Define for $0 < \varepsilon < s$

$$D_\varepsilon := \{(x-\varepsilon, y-\varepsilon) : (x,y) \in D\} .$$

and $(f_n(x), g_n(x)) := R_n\ \hat{R}_n^{\leftarrow}(x,y)$. By Lemma 2.1 for $n \geq n_0$

$$f_n(s) > s - \varepsilon \quad \text{and} \quad g_n(s) > s - \varepsilon .$$

Since $S \subset D$, $(x,y) \in S$ implies $x > s$ or $y > s$ hence $f_n(x) > f_n(s)$ or $g_n(y) > g_n(s)$. It follows that for $n \geq n_0$

$$\begin{aligned}\{(f_n(x), g_n(y)) \ : \ (x,y) \in S\} \subset ((-\infty, f_n(s)] \times (-\infty, g_n(s)])^c \\ \subset ((-\infty, s-\varepsilon] \times (-\infty, s-\varepsilon])^c = D_\varepsilon \ ,\end{aligned}$$

i.e.

$$1_{R_n \ \hat{R}_n^{\leftarrow}(S)} \leq 1_{D_\varepsilon} \ . \tag{12}$$

Next define the measure $\nu^\star$ by

$$\nu^\star := \sum_{n=0}^{\infty} 2^{-n} \tilde{\nu}_n$$

with the convention that $\tilde{\nu}_0 := \nu$. Let h_n be the density of $\tilde{\nu}_n$ with respect to $\nu^\star$. We know from Proposition 2.1

$$\int_S h_n d\nu^\star = \tilde{\nu}_n(S) \to \nu(S) = \int_S h_0 d\nu^\star \quad \text{a.s.} \tag{13}$$

Finally from Lemma 2.1, Proposition 2.1, (12), (13) and Pratt's (1960) lemma

$$\hat{\nu}(S) = \tilde{\nu}_n\left(R_n \ \hat{R}_n^{\leftarrow}(S)\right) = \int 1_{S_n} h_n d\nu^\star \to \int 1_S h_0 d\nu^\star = \nu(S)$$

a.s. hence in probability. □

Proposition 2.3. *Under the conditions of the theorem*

$$\frac{\hat{c}_n}{c_n} \xrightarrow{P} 1 \quad as \quad n \to \infty \ .$$

Proof. Recall $c_n = \sqrt{q_n^2 + r_n^2}$ and $\hat{c}_n = \sqrt{\hat{q}_n^2 + \hat{r}_n^2}$. Now by (4) and (5)

$$\begin{aligned}\hat{q}_n \quad &:= \quad \left(1 + \hat{\gamma}_1 \frac{v_n - \hat{b}_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}_1} \\ &= \quad \left(1 + \hat{\gamma}_1 \left\{\frac{a_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)} \frac{q_n^{\gamma_1} - 1}{\gamma_1} + \frac{b_1\left(\frac{n}{k}\right) - \hat{b}_1\left(\frac{n}{k}\right)}{\hat{a}_1\left(\frac{n}{k}\right)}\right\}\right)^{1/\hat{\gamma}_1} \ .\end{aligned}$$

Then from Lemma 3.1

$$\frac{\hat{q}_n}{q_n} \xrightarrow{P} 1 \ .$$

Similarly

$$\frac{\hat{r}_n}{r_n} \xrightarrow{P} 1 \; .$$

The results follows. ☐

Proposition 2.4. *Under the conditions of the theorem*

$$\hat{\nu}_n \left(\hat{Q}_n \, Q_n^{\leftarrow}(S) \right) \xrightarrow{P} \nu(S) \; .$$

Proof. Invoke a Skorohod construction so that we may pretend that (cf. Condition 2)

$$\hat{\nu}_n(S) \to \nu(S) \; , \frac{\hat{c}_n}{c_n} \to 1 \; ,$$
$$\sqrt{k} \left(\hat{\gamma}_1 - \gamma_i, \frac{\hat{a}_i\left(\frac{n}{k}\right)}{a_i\left(\frac{n}{k}\right)} - 1, \frac{\hat{b}_i\left(\frac{n}{k}\right) - b_i\left(\frac{n}{k}\right)}{a_i\left(\frac{n}{k}\right)} \right) = (O(1), O(1), O(1))$$

for $i = 1, 2$ as $n \to \infty$ almost surely. Then by Lemma 3.1

$$\hat{Q}_n \, Q_n^{\leftarrow}((x,y)) \to ((x,y))$$

for all $(x,y) \in (0,\infty)^2$ a.s. Then from Lemma 2.1

$$1_{\hat{Q}_n \, Q_n^{\leftarrow}(S)}((x,y)) \to 1_S((x,y))$$

for all $(x,y) \in S$ almost surely. The rest of the proof is similar to that of Proposition 2.2. ☐

Proof. (of Theorem 1.1)
We have by Condition 4 as $n \to \infty$

$$p_n = P(C_n) \sim \frac{k}{n c_n} \nu(S) \; .$$

Hence by Proposition 2.3 and Proposition 2.4

$$\frac{\hat{p}_n}{p_n} = \frac{k \, \hat{\nu}_n \left(\hat{Q}_n \, Q_n^{\leftarrow}(S) \right)}{n \, \hat{c}_n \, p_n} \sim \frac{\hat{\nu}_n \left(\hat{Q}_n \, Q_n^{\leftarrow}(S) \right)}{\nu(S)} \frac{c_n}{\hat{c}_n} \xrightarrow{P} 1 \; .$$

☐

3 Result in function space

Example: During surgery the blood pressure of the patient is monitored continuously. It should not go below a certain level and in fact this has never happened in previous similar operations. What is the probability that this happens during surgery of a certain kind?

First a short reminder of extreme value theory in $C[0,1]$ (cf. de Haan and Lin (2001, 2003); Einmahl and Lin (2003)). It is quite similar to the finite-dimensional case. Let $X, X_1, X_2, \ldots$ be i.i.d. continuous stochastic processes on $[0,1]$. We define the maximum of n such processes M_n as an element of $C[0,1]$:

$$M_n(s) := \max_{1 \le i \le n} X_i(s) \text{ for } 0 \le s \le 1 .$$

We consider possible limit distributions of M_n, linearly normalized, as $n \to \infty$. The limit distribution is characterized by two elements: a function $\gamma \in C[0,1]$, the extreme value index function, and a measure ν which is homogeneous of order -1. Write

$$|f| := \sup \{f(s) : s \in [0,1]\} .$$

The domain of attraction condition is: for each Borel set

$$A \subset \{f \in C[0,1] : f \ge 0\}$$

with

$$\nu(\partial A) = 0 , \ \inf \{|f| : f \in A\} > 0 ,$$

$$\lim_{n\to\infty} \frac{n}{k} P\{R_n X \in A\} = \nu(A)$$

for some $k = k(n) \to \infty$, $k/n \to 0$, where for $0 \le s \le 1$

$$R_n X(s) := \left(1 + \gamma(s) \frac{X(s) - b_s\left(\frac{n}{k}\right)}{a_s\left(\frac{n}{k}\right)}\right)^{1/\gamma(s)} .$$

The functions $a_s(n/k) > 0$ and $b_s(n/k)$ are suitable continuous normalizing functions.

Next we suppose that we have a failure set C_n with $P_X(C_n) = O(1/n)$ and a sample $X_1, X_2, \ldots, X_n \in C[0,1]$. We want to estimate $P_X(C_n)$.

Our conditions are quite similar to those in Section 1:

1. The domain of attraction condition.

2. Estimators $\hat{\gamma}(s)$, $\hat{a}_s(n/k)$, $\hat{b}_s(n/k)$ such that with some sequence $k = k(n) \to \infty$, $k(n) = o(n), n \to \infty$,

$$\sup_{0 \le s \le 1} \left(\sqrt{k} \left| \hat{\gamma}(s) - \gamma(s) \right| \bigvee \sqrt{k} \left| \frac{\hat{a}_s\left(\frac{n}{k}\right)}{a_s\left(\frac{n}{k}\right)} - 1 \right| \bigvee \sqrt{k} \left| \frac{\hat{b}_s\left(\frac{n}{k}\right) - b_s\left(\frac{n}{k}\right)}{a_s\left(\frac{n}{k}\right)} \right| \right) = O_P(1) \ .$$

3. C_n is open in $C[0,1]$ and there exists $h_n \in \partial C_n$ such that

$$f \le h_n \Rightarrow f \notin C_n \ .$$

4. (Stability property) We require

$$C_n = \left\{ \left(a_s\left(\frac{n}{k}\right) \frac{(c_n f(s))^{\gamma(s)} - 1}{\gamma(s)} + b_s\left(\frac{n}{k}\right) \right)_{s \in [0,1]} : f \in S \right\}$$

where S is a fixed set in $C[0,1]$ with

- $f \ge 0$ for $f \in S$
- $\nu(\partial S) = 0$ and $\inf\{|f| : f \in S\} > 0$
- $$c_n := \sup_{0 \le s \le 1} \left(1 + \gamma(s) \frac{h_n(s) - b_s\left(\frac{n}{k}\right)}{a_s\left(\frac{n}{k}\right)} \right)^{1/\gamma(s)} .$$

5. Sharpening of Condition 1:

$$\frac{n\ P\left\{R_n(X) \in c_n S\right\}}{k \nu\{c_n S\}} \xrightarrow{P} 1 \ .$$

6. With $\underline{\gamma} := \inf_{0 \le s \le 1} \gamma(s)$:

$$\underline{\gamma} > -\frac{1}{2}$$

and

$$\lim_{n \to \infty} \frac{w_{\underline{\gamma}}(c_n)}{\sqrt{k}} = 0 \ .$$

Finally we define the estimator $\hat{p}_n$ for $p_n := P\{C_n\}$:

$$\hat{p}_n := \frac{1}{n\ \hat{c}_n} \sum_{i=1}^{n} 1_{\left\{\hat{R}_n(X) \in \hat{S}_n\right\}}$$

where

$$\hat{c}_n := \sup_{0\le s\le 1}\left(1+\hat{\gamma}(s)\frac{h_n(s)-\hat{b}_s\left(\frac{n}{k}\right)}{\hat{a}_s\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}(s)} ,$$

$$\hat{R}_n f(s) := \left(1+\hat{\gamma}(s)\frac{f(s)-\hat{b}_s\left(\frac{n}{k}\right)}{\hat{a}_s\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}(s)} \quad \text{for } 0\le s\le 1$$

and

$$\hat{S}_n := \frac{1}{\hat{c}_n}\hat{R}_n(C_n) .$$

Remark 3.1. *Note that $\hat{c}_n$ is not defined if*

$$1+\hat{\gamma}(s)\frac{h_n(s)-\hat{b}_s\left(\frac{n}{k}\right)}{\hat{a}_s\left(\frac{n}{k}\right)} \le 0$$

for some $s\in[0,1]$. However when checking the proof one sees that when $n\to\infty$, the probability that this happens tends to zero.

Theorem 3.1. *Under our conditions*

$$\frac{\hat{p}_n}{p_n} \xrightarrow{P} 1$$

as $n\to\infty$ provided $\nu(S)>0$.

The proof of Theorem 3.1 follows from three lemmas and four propositions. The proofs are very similar to the ones in Section 2 and will be mostly omitted.

Lemma 3.1. *Let G_n be an increasing and invertible mapping: $C[0,1]\to C[0,1]$. Suppose that $\lim_{n\to\infty} G_n f = f$ in $C[0,1]$ for all $f\in C[0,1]$. For an open set O let*

$$O_n := \{G_n f : f\in O\} .$$

Then for all $f\in O$

$$1_{O_n}(f) := 1_{\{f\in O_n\}} \to 1_O(f) := 1_{\{f\in O\}} .$$

Lemma 3.2. *For all $x>0$*

$$\lim_{n\to\infty}\left(1+\gamma(s)\left\{(1+o_1(1))\,\frac{x^{\gamma(s)+o_2(1)}-1}{\gamma(s)+o_2(1)}+o_3(1)\right\}\right)^{1/\gamma(s)} = x ,$$

uniformly for $0\le s\le 1$, provided γ is a continuous function on $[0,1]$ and the o-terms tend to zero uniformly in s.

Lemma 3.3. *For all $x > 0$ and $c_n \to \infty$*

$$\lim_{n\to\infty} \frac{1}{c_n} \left(1 + \gamma_n(s) \left\{ (1 + O_1(\gamma_n(s) - \gamma(s))) \; \frac{(c_n x)^{\gamma(s)} - 1}{\gamma(s)} \right.\right.$$
$$\left.\left. + O_2(\gamma_n(s) - \gamma(s)) \right\}\right)^{1/\gamma_n(s)} = x \; ,$$

uniformly for $0 \le s \le 1$, provided γ_n and γ are continuous functions,

$$\sup_{0\le s\le 1} |\gamma_n(s) - \gamma(s)| = 0$$

and

$$\lim_{n\to\infty} \sup_{0\le s\le 1} |\gamma_n(s) - \gamma(s)| \, c_n^{-\gamma(s)} \int_1^{c_n} s^{\gamma(s)-1} \log s \, ds = 0 \; .$$

Proposition 3.1. *Let S be an open Borel set in $\{f \in C[0,1] : f \ge 0\}$ with $\nu(S) > 0, \nu(\partial S) = 0$ and such that*

$$\inf\{|f| : f \in S\} > 0 \; .$$

Suppose Condition 1 holds. Define

$$\tilde{\nu}_n(S) := \frac{1}{k} \sum_{i=1}^{n} 1_{\{R_n X \in S\}} \; .$$

Then as $n \to \infty$

$$\tilde{\nu}_n(S) \xrightarrow{P} \nu_n(S) \; .$$

Proposition 3.2. *Assume the conditions of Proposition 3.1. Let $\hat{\gamma}(s)$, $\hat{a}_s(n/k)$, $\hat{b}_s(n/k)$ be estimators such that*

$$\sup_{0\le s\le 1} |\hat{\gamma}(s) - \gamma(s)| \xrightarrow{P} 0$$
$$\sup_{0\le s\le 1} \left| \frac{\hat{a}_s\left(\frac{n}{k}\right)}{a_s\left(\frac{n}{k}\right)} - 1 \right| \xrightarrow{P} 0$$
$$\sup_{0\le s\le 1} \left| \frac{\hat{b}_s\left(\frac{n}{k}\right) - b_s\left(\frac{n}{k}\right)}{a_s\left(\frac{n}{k}\right)} \right| \xrightarrow{P} 0 \; .$$

Define

$$\hat{\nu}_n(S) := \frac{1}{k} \sum_{i=1}^{n} 1_{\{\hat{R}_n X \in S\}} \; .$$

Then as $n \to \infty$

$$\hat{\nu}_n(S) \xrightarrow{P} \nu(S) \ .$$

Proof. Invoke a Skorohod construction so that we may assume that in virtue of Lemma 3.2

$$R_n \hat{R}_n^{\leftarrow} \to \text{ Identity } \quad \text{a.s.}$$

Write

$$c := \inf\{|f| : f \in S\} > 0 \ .$$

For all $0 < c_0 < c$ it holds that $f \in S \Rightarrow$ there exists a $s \in [0,1]$ such that $f(s) > c_0$.

Take n_0 such that for $n \geq n_0$

$$R_n \hat{R}_n^{\leftarrow} c_0 > \frac{c_0}{2} \ .$$

Then for each $f \in S$ there exists $s \in [0,1]$ such that

$$R_n \hat{R}_n^{\leftarrow} f(s) > R_n \hat{R}_n^{\leftarrow} c_0 > \frac{c_0}{2} \ .$$

Hence

$$\begin{aligned} \left\{ R_n \hat{R}_n^{\leftarrow} f : f \in S \right\} &\subset \left\{ f : f \leq R_n \hat{R}_n^{\leftarrow} c_0 \right\}^c \\ &\subset \left\{ f : f \leq \frac{c_0}{2} \right\}^c =: D_\varepsilon \ , \end{aligned}$$

i.e.

$$1_{R_n \hat{R}_n^{\leftarrow}(S)} \leq 1_{D_\varepsilon} \ .$$

Now

$$\tilde{\nu}_n(D_\varepsilon) \to \nu(D_\varepsilon)$$

by Proposition 3.1. Hence as in the proof of Proposition 2.2

$$\tilde{\nu}_n \left(R_n \hat{R}_n^{\leftarrow}(S) \right) \to \nu(S)$$

almost surely hence in probability. □

Proposition 3.3. *Under the conditions of the theorem*

$$\frac{\hat{c}_n}{c_n} \xrightarrow{P} 1$$

as $n \to \infty$*.*

Proof. Write for $0 \le s \le 1$

$$r_n(s) := \left(1 + \gamma(s) \frac{h_n(s) - b_s\left(\frac{n}{k}\right)}{a_s\left(\frac{n}{k}\right)}\right)^{1/\gamma(s)} .$$

Then

$$\begin{aligned}
\hat{c}_n &= \sup_{0 \le s \le 1} \left(1 + \hat{\gamma}(s) \frac{h_n(s) - \hat{b}_s\left(\frac{n}{k}\right)}{\hat{a}_s\left(\frac{n}{k}\right)}\right)^{1/\hat{\gamma}(s)} \\
&= \sup_{0 \le s \le 1} r_n(s) \left\{ \frac{1}{r_n(s)} \left(1 + \hat{\gamma}(s) \left[\frac{a_s\left(\frac{n}{k}\right)}{\hat{a}_s\left(\frac{n}{k}\right)} \frac{(r_n(s))^{\gamma(s)} - 1}{\gamma(s)} \right.\right.\right. \\
&\quad \left.\left.\left. + \frac{b_s\left(\frac{n}{k}\right) - \hat{b}_s\left(\frac{n}{k}\right)}{\hat{a}_s\left(\frac{n}{k}\right)} \right]\right)^{1/\hat{\gamma}(s)} \right\} .
\end{aligned}$$

Hence, since the expression inside the curly brackets tends to one in probability uniformly in s by Lemma 3.3, we have

$$\frac{\hat{c}_n}{c_n} = \frac{\hat{c}_n}{\sup_{0 \le s \le 1} r_n(s)} \xrightarrow{P} 1 .$$

□

Proposition 3.4. *Under the conditions of the theorem*

$$\hat{\nu}_n\left(\frac{1}{\hat{c}_n} \hat{R}_n \, R_n^{\leftarrow}(c_n S)\right) \xrightarrow{P} \nu_n(S) .$$

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