Extreme Value Theory: An Introduction
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This document contains corrections of errors or mistakes found in the book after publication. We acknowledge the contribution of the readers.

We also give extensions of some of the material given in the book.

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Chapter 1: Limit Distributions and Domains of Attraction

Page **12**, *line* **14**, *correction*: $b'_n = (2\log n)^{1/2} - \frac{\log\log n + \log(4\pi)}{2(2\log n)^{1/2}}$.

Chapter 2: Extreme and Intermediate Order Statistics

Page **42**, *line* **10**, *correction*: $\sqrt{k}\left(\frac{nU_{k+1,n}}{k} - 1\right)$.

Chapter 3: Estimation of the Extreme Value Index and Testing

Page **77**, *line* **6**, *correction*: $\left(\left|\frac{\lambda}{\rho c_2}\right| \gamma c_1^{1-\rho}\right)^{2/(1-2\rho)} n^{-2\rho/(1-2\rho)}$.

Page **111**, *line* **7**, *correction*: Then (3.6.5) multiplied by $f(t)$ becomes (...)

line **10**, *correction*: and (3.6.6) multiplied by $f(t)$ becomes

Chapter 4: Extreme Quantile and Tail Estimation

Page **128**, *Fig.4.2(b) label vertical axis*, *correction*: $\log(1+\gamma x)/\gamma$

Page **134**, *line* **-12**, *addendum*: **Theorem 4.3.1 (de Haan and Rootzén (1993))**

Page **138**, *line* **10**, *addendum*: **Theorem 4.3.8 (Dijk and de Haan (1992))**

Page **148**, *Sect. 4.6.1. Simulations*, *correction*: (...) $p_n = 1/(n\log n) = 1.448 \times 10^{-4}$ (...) $n = 1\,000$ (yet $p_n = 1.086 \times 10^{-4}$ for the Cauchy distribution, which indeed corresponds to 2931.7 in Fig. 4.4)

Table 4.1. Sea level data: 95% asymptotic confidence intervals for quantile.

k	100		200		300	
Moment	(350.,	1029.)	(419.,	1015.)	(454.,	988.)
PWM	(299.,	1017.)	(372.,	1049.)	(411.,	1018.)

Page **151**, *Table 4.1*, *correction*:

Table 4.2, *clarification*: "3rd Quantile" is the same as "0.75 Quantile".

Chapter 6: Basic Theory (in higher dimensional space)

Page **213**, *Theorem 6.1.9*, *correction*:
Theorem 6.1.9 For any Borel set $A \subset \mathbb{R}_+^2$ with $\inf_{(x,y)\in A} \max(x,y) > 0$ and any $a > 0$, (...)

Page **217**, *Definition 6.1.13*, *reformulation*:
Definition 6.1.13 A distribution function G is called *max-stable* if there are constants $A_n > 0$, $C_n > 0$, B_n, and D_n such that for all x, y and $n = 1, 2, \ldots$,

$$G^n(A_n x + B_n, C_n y + D_n) = G(x,y).$$

Any distribution function G satisfying (6.1.25) is max-stable (cf. (6.1.28)), and also $G(\alpha x + \beta, \gamma y + \delta)$ where $\alpha > 0$, $\gamma > 0$, β, and δ are arbitrary real constants. Since any max-stable distribution is in the class of limit distributions for (6.1.1), we get all the max-stable distributions (i.e. the class of limit distribution functions G) this way. The class of limit distribution functions G_0 in (6.1.10) is called the class of *simple max-stable distributions*, "simple" meaning that the marginal distributions are fixed as follows: $G_0(x,\infty) = G_0(\infty, x) = \exp(-1/x)$, $x > 0$.

Page **231**, *Exercise 6.2*, *reformulation*:
6.2. (...) converges to

$$g(x,y) := 2^{-1}\log(\lambda/(4\pi)) - (4\lambda)^{-1} - \lambda 4^{-1}(x-y)^2 - 2^{-1}(x+y)$$

for $x, y \in \mathbb{R}$. Conclude that

$$\lim_{n\to\infty} n\left(1 - F_n(a_n x + b_n, a_n y + b_n)\right) = \iint_{\{s>x\}\cup\{t>y\}} g(s,t)\, ds\, dt,$$

for $x, y \in \mathbb{R}$.

Exercise 6.3 (d), *addendum*: with $tA^{-1} := \left\{ \left(\frac{t}{x}, \frac{t}{y}\right),\ (x,y) \in A \right\}$

Page **232**, *Exercise 6.6, addendum*:

6.6. (...) $H'(\theta) = r^3 q(\theta r, r(1-\theta)) = q(\theta, (1-\theta))$ (cf. Coles and Tawn (1991)).

Exercise 6.11, reformulation:

6.11. Let $(V_1, V_2, \ldots, V_d)$ be independent and identically distributed random variables with distribution function $\exp -(1/x)$, $x > 0$. Let $\{r_{i,j}\}_{i=1,2;j=1,\ldots,d}$ be a matrix with positive entries. Show that the random vector $(\vee_{j=1}^d r_{1,j} V_j, \vee_{j=1}^d r_{2,j} V_j)$ has a simple max-stable distribution. Find the distribution function. Show that any two dimensional distribution function with Fréchet marginals can be obtained as a limit of elements in this class.

Exercise 6.12, correction: $\exp - \left(\left(\frac{x}{\lambda_1} \wedge \frac{y}{\nu_1} \right)^{-\alpha} + \left(\frac{x}{\lambda_2} \wedge \frac{y}{\nu_2} \right)^{-\alpha} \right)$

New Exercise:

6.14. Let (X, Y) have a standard spherically symmetric Cauchy distribution. Show that the probability distribution of $(|X|, |Y|)$ is in the domain of attraction of an extreme value distribution with uniform spectral measure Ψ. Show that the probability distribution of (X, Y) is also in a domain of attraction. Find the limit distribution.

Chapter 7: Estimation of the Dependence Structure

Page **268**, *Exercise 7.3, correction*:

$$EW(x_1, \ldots, x_d) W(y_1, \ldots, y_d) = \mu \left(R(x_1, \ldots, x_d) \cap R(y_1, \ldots, y_d) \right)$$

Page **269**, *Exercise 7.4, correction*:

7.4. (...), and N indicates a normal probability distribution.

Chapter 9: Basic Theory in $C[0, 1]$

Page **306**, *lines* **7–19**, *correction*: These lines should be indented, as they belong to part (2) of the proof.

line **11**, *addendum*: First we note that this assumption implies

Page **308**, *Example 9.4.6, reformulation*:

Example 9.4.6 A nice example of a simple max-stable process has already been given by Brown and Resnick (1977). Let W^* be two-sided Brownian motion:

$$W^*(s) := \begin{cases} W^+(s) \,, & s \geq 0 \\ W^-(-s) \,, & s < 0 \end{cases}$$

where W^+ and W^- are independent Brownian motions. For the independent and identically distributed processes $\{V_i\}_{i=1}^{\infty}$ of Corollary 9.4.5 take

$$\{V_i(s)\}_{s\in\mathbb{R}} := \left\{e^{W_i^*(s)-|s|/2}\right\}_{s\in\mathbb{R}} ,$$

where the W_i^* are independent Brownian motions, i.e.,

$$\{\eta(s)\}_{s\in\mathbb{R}} := \left\{\bigvee_{i=1}^{\infty} Z_i\, e^{W_i^*(s)-|s|/2}\right\}_{s\in\mathbb{R}} ,$$

where $\{Z_i\}_{i=1}^{\infty}$ is a realization of a Poisson point process on $(0,\infty]$ with mean measure dr/r^2 and independent of $\{W_i^*\}_{i=1}^{\infty}$. The process η is stationary (cf. Section 9.8 below).

Page **311**, *line* **-5**, *addendum*: **Theorem 9.5.1 (de Haan and Lin (2001))**

Page **315**, *line* **3**, *correction*: (...) $\sup_{0\le s\le 1} f_s(t) = c$ for all $t \in [0,1]$.

line **5**, *addendum*: **Theorem 9.6.1 (Resnick and Roy (1991) and de Haan(1984))**

Section 9.6.2 Stationarity, *addendum*: An extension of the results of this section to $t \in \mathbb{R}$ is in the Appendix bellow.

Page **323**, *Section 9.8 Two Examples*, *reformulation*:
9.8 Two Examples Let us go back to Example 9.4.6 of Section 9.4 (with the aforementioned correction). Let W^* be two-sided Brownian motion,

$$W^*(s) := \begin{cases} W^+(s)\,, & s \ge 0 \\ W^-(-s)\,, & s < 0 \end{cases}$$

where W^+ and W^- are independent Brownian motions. Then consider

$$\{\eta(s)\}_{s\in\mathbb{R}} := \left\{\bigvee_{i=1}^{\infty} Z_i e^{W_i^*(s)-|s|/2}\right\}_{s\in\mathbb{R}} \tag{1}$$

where $\{Z_i\}_{i=1}^{\infty}$ is a realization of a Poisson point process on $(0,\infty]$ with mean measure dr/r^2 and independently, $\{W_i^*\}_{i=1}^{\infty}$ is a sequence of independent two-sided Brownian motions.

Then, to check that the process is stationary just follow the same calculations from pages 323 to 325 with W replaced by W^*.

Page **324**, *line* **-5**, *correction*:

$$F_2'(t) = \frac{1}{t\sqrt{s_2}}\,\phi\left(\frac{\sqrt{s_2}}{2} + \frac{y}{\sqrt{s_2}} + \frac{\log t}{\sqrt{s_2}}\right)$$

Page **326,** *Example 9.8.1, reformulation*:

Example 9.8.1 Let Y be a random variable with distribution function $1-1/x$, $x \geq 1$. Let W^* be two-sided Brownian motion,

$$W^*(s) := \begin{cases} W^+(s) \ , & s \geq 0 \\ W^-(-s) \ , & s < 0 \end{cases}$$

where W^+ and W^- are independent Brownian motions. Let Y and W^* be independent. Consider the process (... follow the rest of the example with W replaced by W^*).

line **-14,** *correction*: (...) For $a > 1$ (...)

Example 9.8.2, reformulation:

Example 9.8.2 (Extension of Brown and Resnick (1977)) Let $\{X(s)\}_{s\in\mathbb{R}}$ be a Ornstein-Uhlenbeck process, i.e.,

$$\begin{aligned} X(s) &= 1_{\{s\geq 0\}} e^{-s/2}\left(N + \int_0^s e^{u/2} dW^+(u)\right) \\ &\quad + 1_{\{s<0\}} e^{s/2}\left(N + \int_0^{-s} e^{u/2} dW^-(u)\right) \end{aligned}$$

with N, W^+ and W^- independent, N a standard normal random variable and W^+ and W^- standard Brownian motions. Since for $s \neq t$ the random vector $(X(s), X(t))$ is multivariate normal with correlation coefficient less than one, Example 6.2.6 tells us that, relation (9.5.1) can not hold for any max-stable process in $C[0,1]$: since Y has continuous sample paths, $Y(s)$ and $Y(t)$ can not be independent. Hence we compress space in order to create more dependence, i.e., we consider the convergence of

$$\left\{ \bigvee_{i=1}^n b_n \left(X_i\left(\frac{s}{b_n^2}\right) - b_n \right) \right\}_{s\in\mathbb{R}} \tag{2}$$

in $C[-s_0, s_0]$ for arbitrary $s_0 > 0$, where $X_1, X_2, \ldots$ are independent and identically distributed copies of X and the b_n are the appropriate normalizing constants for the standard one-dimensional normal distribution, e.g., $b_n = (2\log n - \log\log n - \log(4\pi))^{1/2}$ (cf. Example 1.1.7). Then

$$b_n\left(X\left(\frac{s}{b_n^2}\right) - b_n\right) =$$

$$e^{-|s|/(2b_n^2)}\left(b_n(N - b_n) + b_n\int_0^{|s|/b_n^2} e^{u/2} dW^{\pm}(u) + \left(1 - e^{|s|/(2b_n^2)}\right) b_n^2\right)$$

where $W^{\pm}(s)$ is $W^+(s)$ for $s \geq 0$ and $W^-(s)$ for $s < 0$. Note that uniformly for $|s| \leq s_0$,

$$e^{-|s|/(2b_n^2)} = 1 + O\left(\frac{1}{b_n^2}\right) .$$

Further, since $e^{u/2} = 1 + O\left(1/b_n^2\right)$ for $|u| < s_0/b_n^2$,

$$b_n \int_0^{|s|/b_n^2} e^{u/2} dW^{\pm}(u) = \left(1 + O\left(\frac{1}{b_n^2}\right)\right) b_n W^{\pm}\left(\frac{|s|}{b_n^2}\right) .$$

Finally, for $|s| \le s_0$,

$$\left(1 - e^{|s|/(2b_n^2)}\right) b_n^2 = -\frac{|s|}{2} + O\left(\frac{1}{b_n^2}\right) .$$

It follows that

$$\begin{aligned} & b_n \left(X\left(\frac{s}{b_n^2}\right) - b_n \right) \\ & \quad = \left(1 + O\left(\frac{1}{b_n^2}\right)\right) \left(b_n (N - b_n) + b_n W^{\pm}\left(\frac{|s|}{b_n^2}\right) - \frac{|s|}{2} \right) + O\left(\frac{1}{b_n^2}\right) . \end{aligned}$$

We write $W^{\star}(|s|) := b_n W^{\pm}\left(|s|/b_n^2\right)$. Then $W^{\star}$ is also Brownian motion. We have

$$\begin{aligned} & \left\{ \bigvee_{i=1}^{n} b_n \left(X_i\left(\frac{s}{b_n^2}\right) - b_n \right) \right\}_{s\in\mathbb{R}} \\ & = \left(1 + O\left(\frac{1}{b_n^2}\right)\right) \left\{ \bigvee_{i=1}^{n} \left(b_n (N_i - b_n) + W_i^{\star}(|s|) \right) - \frac{|s|}{2} \right\}_{s\in\mathbb{R}} + O\left(\frac{1}{b_n^2}\right) . \end{aligned}$$

Hence the limit of (2) is the same as that of

$$\left\{ \bigvee_{i=1}^{n} \left(b_n (N_i - b_n) + W_i^{\star}(|s|) \right) - \frac{|s|}{2} \right\}_{s\in\mathbb{R}} . \tag{3}$$

The rest of the proof runs as in the previous example.

One finds that the sequence of processes (3) converges weakly in $C[-s_0, s_0]$, hence in $C(\mathbb{R})$, to

$$\left\{ \bigvee_{i=1}^{\infty} \left(\log Z_i + W_i^{\star}(|s|) \right) - \frac{|s|}{2} \right\}_{s\in\mathbb{R}}$$

with $\{Z_i\}$ the point process from (9.8.1).

Note that the point process $\{Z_i\}$ and the random processes $W_i^{\star}$ are independent.

Page **328,** *Exercise 9.5, correction***:**

9.5. (...) V is a continuous stochastic process independent of Y (...)

$$P(\xi(s) > x) = \frac{1}{x} \int_0^x P(V(s) > u)\, du , \quad x > 0 .$$

(...)

Chapter 10: Estimation in $C[0,1]$

Page **332**, *line* **-2**, *addendum*: **Theorem 10.2.1 (de Haan and Lin (2003))**

Page **339**, *line* **-3**, *addendum*: **Theorem 10.4.1 (de Haan and Lin (2003))**

Page **341**, *line* **-3**, *correction*:

$$= \left(\frac{n}{k}\frac{1}{\zeta_{n-k,n}(s)}\right)^{\mu(s)} \int_{(k/n)\zeta_{n-k,n}(s)}^{\infty} (1 - G_{n,s}(x))\, x^{\mu(s)-1}\, dx$$

Page **352**, *line* **6**, *correction*:

$$\tilde{\nu}_n(S) \xrightarrow{P} \nu(S)\ .$$

Appendix B: Regular Variation and Extensions

Page **366**, *line* **5**, *correction*: $\exp\left(\int_{t_0}^{t} a(v)\ \frac{dv}{v}\right)$

Page **380**, *line* **9**, *correction*:

$$(1-\delta_2)\frac{1-x^{-\delta_1}}{-\delta_1} - \delta_2 < \frac{f(tx)-f(t)}{a(t)} < (1+\delta_2)\frac{x^{\delta_1}-1}{\delta_1} + \delta_2\ .$$

Page **381**, *line* **-8**, *addendum*:
(4) From part (3) of the present proposition it follows (...)

Further and Updated References

S. Coles and J. Tawn: Modeling extreme multivariate events. *J. Royal Statist. Soc. Ser. B* **53**, 277–392 (1991).

V. Dijk and L. de Haan: On the estimation of the exceedance probability of a high level. In: P.K. Sen and I.A. Salama (eds) *Order Statistics and Non-Parametrics: Theory and Applications*, Elsevier, Amsterdam, 79–92 (1992).

L. de Haan and T. Lin: On convergence towards an extreme value distribution in C[0,1]. *Ann. Prob.* **29**, 467–483 (2001).

L. de Haan and T. Lin: Weak consistency of extreme value estimators in C[0,1]. *Ann. Statist.* **31**, 1996–2012 (2003).

L. de Haan and T.T. Pereira: Spatial Extremes: Models for the stationary case. *Ann. Statist.* **34**, 146–168 (2006).

L. de Haan and H. Rootzén: On the estimation of high quantiles. *J. Statist. Plann. Inference* **35**, 1–13 (1993).

Appendix: Extension of Section 9.6.2 Stationarity

Corollary 9.6.7.A *Let $\{(Z_i, T_i)\}_{i=1}^{\infty}$ be a realization of a Poisson point process on $(0, \infty] \times \mathbb{R}$ with mean measure $(dr/r^2) \times d\lambda$ (λ Lebesgue measure). If η is a simple max-stable process in $C^+(\mathbb{R})$, then there exists a family of functions $f_s(t)$ ($s, t \in \mathbb{R}$) with*

1. *for each $t \in \mathbb{R}$ we have a non-negative continuous function $f_s(t) : \mathbb{R} \to [0, \infty)$,*
2. *for each $s \in \mathbb{R}$*
$$\int_{-\infty}^{+\infty} f_s(t)\, dt = 1 \,, \tag{4}$$
3. *for each compact interval $I \subset \mathbb{R}$*
$$\int_{-\infty}^{+\infty} \sup_{s \in I} f_s(t)\, dt < \infty \,,$$

such that
$$\{\eta(s)\}_{s \in \mathbb{R}} \stackrel{d}{=} \left\{ \bigvee_{i=1}^{\infty} Z_i\, f_s(T_i) \right\}_{s \in \mathbb{R}} \,. \tag{5}$$
Conversely every process of the form exhibited at the right-hand side of (5) with the stated conditions, is a simple max-stable process in $C^+(\mathbb{R})$.

Proof. Let H be a probability distribution function with a density H', that is positive for all real x. With the functions f_s from Theorem 9.6.7 define the functions $\tilde{f}_s(t) := f_s(H(t))\, H'(t)$. Since for any $s_1, \ldots, s_d \in \mathbb{R}$ and $x_1, \ldots, x_d \in \mathbb{R}$ positive,
$$\int_{-\infty}^{+\infty} \max_{1 \le i \le d} \frac{\tilde{f}_{s_i}(t)}{x_i}\, dt = \int_0^1 \max_{1 \le i \le d} \frac{f_{s_i}(t)}{x_i}\, dt \,,$$
the representation of the corollary follows easily from that of Theorem 9.6.7.

Definition 9.6.9.A *A mapping Φ from L_1^+ (the non-negative integrable functions on $\mathbb{R}$) to L_1^+ is called a* piston *if for $h \in L_1^+$*
$$\Phi(h(t)) = r(t) h(H(t))$$
with H a one-to-one measurable mapping from $\mathbb{R}$ to $\mathbb{R}$ and r a positive measurable function, such that for every $h \in L_1^+$
$$\int_{-\infty}^{+\infty} \Phi(h(t))\, dt = \int_{-\infty}^{+\infty} h(t)\, dt \,.$$

Theorem 9.6.10.A *Let* $\{(Z_i, T_i)\}_{i=1}^{\infty}$ *be a realization of a Poisson process on* $(0,\infty] \times \mathbb{R}$ *with mean measure* $(dr/r^2) \times d\lambda$ *(*λ *Lebesgue measure).*

If the stochastic process $\{\eta(s)\}_{s\in\mathbb{R}}$ *is simple max-stable, stricly stationary and continuous a.s., then there is a function* h *in* L_1^+ *with* $\int_{-\infty}^{+\infty} h(t)dt = 1$ *and a continuous group of pistons* $\{\Phi_s\}_{s\in\mathbb{R}}$ *(continuous i.e.,* $\Phi_{s_n}(h(t)) \to \Phi_s(h(t))$ *as* $s_n \to s$ *for almost all* $t \in \mathbb{R}$*) with*

$$\int_{-\infty}^{+\infty} \sup_{s\in I} \Phi_s\left(h(t)\right)\ dt < \infty$$

for each compact interval I*, such that*

$$\{\eta(s)\}_{s\in\mathbb{R}} \stackrel{d}{=} \left\{\bigvee_{i=1}^{\infty} Z_i\ \Phi_s\left(h(T_i)\right)\right\}_{s\in\mathbb{R}} . \tag{6}$$

Conversely every stochastic process of the form exhibited at the right-hand side of (6) *with the stated conditions, is simple max-stable, strictly stationary and a.s. continuous.*

Proof. Just replace everywhere in the proof of Theorem 9.6.10, $t \in [0,1]$ by $t \in \mathbb{R}$.